THE DILOGARITHMIC CENTRAL EXTENSION OF THE PTOLEMY-THOMPSON GROUP VIA THE KASHAEV QUANTIZATION

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Abstract. Quantization of the Teichmüller space of a surface yields projective representations of the mapping class group and hence central extensions of it. Using the Chekhov-Fock quantization (or the Fock-Goncharov formulation) of the universal Teichmüller space, Funar and Sergiescu computed the central extension $\widehat{T}$ of the Ptolemy-Thompson group T, which is a universal version of the mapping class groups, and this extension has the extension class 12 times the Euler class χ. We compute the central extension of T using the Kashaev quantization, and show that its extension class is 6χ. We also give a direct graphical proof of the isomorphism between this central extension and the group $T^\sharp_{ab}$, the relative abelianization of the braided Ptolemy-Thompon group $T^\sharp$ of Funar and Kapoudjian. The conjecture that the representation-theoretic construction of the Kashaev quantization by Igor Frenkel and the author may lead to a construction of a representation of $T^\sharp$ is also stated.

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1. Introduction

It is known that quantization of the Teichmüller space of a surface with boundary and punctures, first achieved in late 90's by Kashaev [Kas1] and independently by Chekhov-Fock [Fo] [CFo] (then later extended to the more general framework of the cluster algebras by Fock-Goncharov [FoG]), yields certain projective representations of the mapping class group of the surface. The main ingredient of both the constructions is a special function called 'quantum dilogarithm', which has received much attention since Faddeev and Kashaev introduced it in the context of the quantum groups around 1995 [FaKas] [Fa] (but whose formula (3.21) was already known to Barnes [Ba] in 1901). There is a universal setting for this construction, namely quantization of the universal Teichmüller space ([P2]), which can be thought of as the Teichmüller space of the open unit disc $\mathbb{D}$ with certain boundary behavior, or of the closed unit disc with countable number of distinguished points on the boundary (in the sense of Teichmüller theory on surfaces with boundary with distinguished points on the boundary; see [P3]). Quantization requires the choice of a coordinate system on the Teichmüller space, which depends on the choice of a certain infinite triangulation of the surface $\mathbb{D}$, which is also called a 'tessellation' of $\mathbb{D}$. Then the mapping class group on the surface can be described in terms of the changes of tessellations, or the 'flips' along the edges of tessellations. Due to its isomorphism to one version of Thompson's groups, this (universal) mapping class group is dubbed 'Ptolemy-Thompson group' by Funar, Kapoudjian, Sergiescu and collaborators [FuKap2] [FuS] [FuKapS], denoted by T by them, and denoted by G_{mark} in the present paper (the subscript 'mark' stands for the 'marked tessellations'): this group has the following finite presentation (2.9) (due to Lochak-Schneps [LoSc]):

$$T \cong G_{mark} = \left\langle \alpha, \beta \;\middle|\; \begin{array}{l} (\beta\alpha)^5 = \alpha^4 = \beta^3 = 1, \\ \left[\beta\alpha\beta,\, \alpha^2\beta\alpha\beta\alpha^2\right] = \left[\beta\alpha\beta,\, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2\right] = 1 \end{array} \right\rangle . \tag{1.1}$$

The Kashaev quantization and the Chekhov-Fock quantization provide projective representations ρ^{Kash} and ρ^{CF} of this group G_{mark} respectively, by unitary operators on an infinite dimensional Hilbert space.

It is well known that a projective representation of a group leads to a central extension of the group (we describe a slightly general version of this procedure in §4.1). Funar and Sergiescu [FuS] classified the presentations of all possible central extensions of $T \cong G_{mark}$ by $\mathbb{Z}$ (as written in Theorem 4.7 in the present paper); let $T_{n,p,q,r}$ be the group presented by the generators $\bar\alpha$, $\bar\beta$, z and the relations

$$\begin{array}{l} (\bar\beta\bar\alpha)^5 = z^n, \quad \bar\alpha^4 = z^p, \quad \bar\beta^3 = z^q, \\ [\, \bar\beta\bar\alpha\bar\beta,\, \bar\alpha^2\bar\beta\bar\alpha\bar\beta\bar\alpha^2 \,] = z^r, \quad [\, \bar\beta\bar\alpha\bar\beta,\, \bar\alpha^2\bar\beta\bar\alpha^2\bar\beta\bar\alpha\bar\beta\bar\alpha^2\bar\beta^2\bar\alpha^2 \,] = 1, \quad [\, \bar\alpha, z \,] = [\, \bar\beta, z \,] = 1. \end{array} \tag{1.2}$$

Then each central extension of the Ptolemy-Thompson group $T \cong G_{mark}$ by $\mathbb{Z}$ is isomorphic to $T_{n,p,q,r}$ for some $n,p,q,r \in \mathbb{Z}$. Moreover in [FuS], the class $c_{T_{n,p,q,r}} \in H^2(T)$ of the extension

$T_{n,p,q,r}$ is computed to be

$$c_{T_{n,p,q,r}} = (12n - 15p - 20q - 60r)\chi + r\alpha, \tag{1.3}$$

where $\alpha, \chi \in H^2(T)$ are the so-called discrete Godbillon-Vey class and the Euler class, respectively. One of Funar-Sergiescu's main results is the computation of a presentation of the central extension $\widehat{T} = \widehat{G}^{CF}_{mark}$ (which they called the 'dilogarithmic (central) extension') of $T \cong G_{mark}$ induced by Chekhov-Fock's projective representation ρ^{CF}:

$$\widehat{G}^{CF}_{mark} \cong T_{1,0,0,0}, \quad \text{which corresponds to the class} \quad c_{CF} = 12\chi \in H^2(G_{mark}). \tag{1.4}$$

The main result of the present paper is the computation of a presentation of the central extension $\widehat{G}^{Kash}_{mark}$ of G_{mark} induced by Kashaev's projective representation ρ^{Kash}:

$$\widehat{G}^{Kash}_{mark} \cong T_{3,2,0,0}, \quad \text{which corresponds to the class} \quad c_{CF} = 6\chi \in H^2(G_{mark}). \tag{1.5}$$

This discrepancy of the two central extension classes induced by ρ^{Kash} and ρ^{CF} may shed light on the way to explaining the relationship between the two quantizations of Kashaev and Chekhov-Fock, which is still not completely clear. See Guo-Liu [GuLi] for a purely algebraic relationship between the two constructions, which doesn't capture the above discrepancy; see the remarks at the end of §4.1 of the present paper.

We first give an 'algebraic' proof of (1.5). For this, we need to establish a bridge between two different decorations on tessellations (enhanced tessellations). A *marked tessellation* is a tessellation with the choice of a distinguished oriented edge, and a *dotted tessellation* is a tessellation with the choice of a distinguished corner for each triangle. We give a natural way of assigning a dotted tessellation to a marked tessellation, which induces an assignment of a change of dotted tessellations to a change of marked tessellations. The Ptolemy-Thompson group $T \cong G_{mark}$ is the group of changes of marked tessellations (induced by certain homeomorphisms of $\mathbb{D}$ to itself) and can also be viewed as a mapping class group in a certain sense, while *the Kashaev group* G_{dot} is the *formal* group of changes of dotted tessellations. The above correspondence gives a natural injective group homomorphism $G_{mark} \to G_{dot}$. The Chekhov-Fock quantization provides a projective representation of G_{mark}, while the Kashaev quantization a priori provides that of G_{dot}. By the above homomorphism, we can turn Kashaev's projective representation of G_{dot} into that of G_{mark}. So, each generator α and β of G_{mark} is represented via Kashaev's operators, and we compute the relations in (1.1) using these operators. This yields the lifted relations for the sought-for central extension $\widehat{G}^{Kash}_{mark}$.

In fact, the central extensions $T_{1,0,0,0}$ and $T_{3,2,0,0}$ of T are special, because they have 'geometric' meanings. For the moment, as done in [FuKap2] (and in the other works of Funar, Kapoudjian, Sergiescu, and collaborators), let T be (a certain version of) the mapping class group of the 'ribbon tree', which is the planar surface obtained by thickening the infinite trivalent graph (or the infinite binary tree). By introducing infinitely many punctures on the ribbon tree in certain two different ways, we get the corresponding mapping class groups T^* and $T^\sharp$ ([FuKap2]), which are extensions of T by the infinite braid group B_∞, which is the braid group associated to the infinite number of punctures. Then, the abelianization homomorphism $B_\infty \to H_1(B_\infty) = \mathbb{Z}$ induces the 'relative abelianizations' T^*_{ab} and $T^\sharp_{ab}$, which are central extensions of T by $\mathbb{Z}$. Funar and Kapoudjian computed their presentations in [FuKap2]:

$$T^*_{ab} \cong T_{1,0,0,0}, \quad T^\sharp_{ab} \cong T_{3,2,0,0}, \tag{1.6}$$

hence we get $T^*_{ab} \cong \widehat{G}^{CF}_{mark}$ (as pointed out in [FuS]) and $T^\sharp_{ab} \cong \widehat{G}^{Kash}_{mark}$. It is quite interesting to see that the projective representations of Chekhov-Fock and of Kashaev precisely capture

these topological information about the two different kinds of punctures (or vice versa). One can further ask (as raised to the author by Funar [Fu]) if we can directly relate $\widehat{G}^{Kash}_{mark}$ and $T^\sharp_{ab}$, without consulting their explicit presentations $T_{3,2,0,0}$. This turns out to be possible, and a 'graphical' proof of

$$\widehat{G}^{Kash}_{mark} \cong T^\sharp_{ab} \tag{1.7}$$

without computing their explicit presentations is also done in the present paper. However, we need to translate the ribbon tree formulation to our tessellation formulation.

We first observe that the ribbon tree model of Funar and collaborators is dual to (equivalent to) the tessellation model, and we thus transfer this idea of infinite punctures to the tessellation model. We introduce the two different ways of having infinite number of punctures on the unit disc, either one puncture on every edge of a tessellation, or one puncture on the interior of every triangle of a tessellation; we mainly study the latter case, indicated by the superscript $\sharp$. We define the tessellations of this infinitely-punctured unit disc, and the two decorated (enhanced) versions of them; the marked version and the dotted version. The groups $G^\sharp_{mark}$ and $G^\sharp_{dot}$ of changes of these new decorated punctured tessellations are defined, and they are found to be extensions of G_{mark} and G_{dot} by the infinite braid group B_∞, which is the braid group associated to the infinite number of punctures. By abelianizing B_∞, we obtain the 'relative abelianizations' $(G^\sharp_{mark})_{ab}$ and $(G^\sharp_{dot})_{ab}$, which are central extensions of G_{mark} and G_{dot}. The natural maps $G^\sharp_{mark} \to G^\sharp_{dot}$ and $(G^\sharp_{mark})_{ab} \to (G^\sharp_{dot})_{ab}$ are also studied, as analogues of the non-punctured case. It turns out that the relations among the generators of the group $(G^\sharp_{dot})_{ab}$, which can be viewed as 'a geometric central extension of the Kashaev group' and whose relations are easily checked by pictures, are exactly same as the relations satisfied by the Kashaev operators for the generators of G_{dot} (coming from the Kashaev quantization). Via the natural relationships between G_{mark} and G_{dot} and between $(G^\sharp_{mark})_{ab}$ and $(G^\sharp_{dot})_{ab}$, we are able to identify $(G^\sharp_{mark})_{ab}$ and the group generated by the Kashaev operators for the generators of G_{mark}, namely $\widehat{G}^{Kash}_{mark}$. Then, since we have $T^\sharp_{ab} \cong (G^\sharp_{mark})_{ab}$ from the duality between the ribbon tree model and the tessellation model, we get (1.7).

Our algebraic proof of (1.5) involves the extensive computation of the (lifted) α, β-relations of the Ptolemy-Thompson group $G_{mark} \cong T$, while our graphical proof of (1.7) only involves the very simple checking of the four main relations of the Kashaev group G_{dot} by means of the pictures for the punctured tessellations. However, (1.7) itself cannot be a replacement for the proof of (1.5) (which we do need, in order to compare with (1.4)). In order to get (1.5), in addition to (1.7) we also need the result $T^\sharp_{ab} \cong T_{3,2,0,0}$ of [FuKap2] (see (1.6)) for the presentation of $T^\sharp_{ab}$. Funar-Kapoudjian's proof of $T^\sharp_{ab} \cong T_{3,2,0,0}$ ([FuKap2]) involves extensive topological computation of the (lifted) α, β-relations. Hence, our algebraic proof of (1.5) together with our identification (1.7) can be viewed as an algebraic replacement of Funar-Kapoudjian's topological proof of $T^\sharp_{ab} \cong T_{3,2,0,0}$ ([FuKap2]). Anyways, we still can keep in mind the philosophy (as pointed out by Frenkel [Fr]) that although the Ptolemy-Thompson group $G_{mark} \cong T$ is a more natural group than the (more artificial) Kashaev group G_{dot} which contains G_{mark} as a subgroup, the computations can get much easier for G_{dot} than they are for G_{mark}.

We also state some further directions of research. For example, all the constructions, proofs and the comparison between the two different central extensions that were done in the present paper can be applied to finite type surfaces, too. In fact, Funar and Kashaev [FuKas] computed the extension classes of the central extensions of the mapping class groups of finite type surfaces

using the Kashaev quantization. One may try to repeat this construction for finite type surfaces using the Chekhov-Fock quantization instead, and compare with the result of [FuKas].

Another very interesting open question (suggested by Frenkel [Fr] and Funar [Fu]) is the construction of representations of T^* and/or $T^\sharp$, which are extensions of T by the full infinite braid group B_∞, instead of their relative abelianizations T^*_{ab} and $T^\sharp_{ab}$. As the result of Lochak and Schneps [LoSc] suggests, this then may lead to a representation of the Grothendieck-Teichmüller group $\widehat{GT}$, and hence of the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, on the quantum universal Teichmüller space (which is a question raised in [Fr]). To construct a representation of T^* or $T^\sharp$, we in particular should seek for how to faithfully represent the braids. Igor Frenkel [Fr] suggested a natural candidate for the construction, using the recent result of him and the author [FrKi], which realized the Kashaev quantization in terms of the representation theory of a rather basic Hopf algebra, the 'modular double' of the 'quantum plane algebra'. His conjecture is that the braids can be represented by some analogues of the 'R-matrices'. Before trying this, one may also look for an interpretation of α and β in terms of the representation theory of this Hopf algebra.

The present paper is made to be essentially self-contained, to avoid the confusion on the different notations used by different authors. Readers can consult the table of contents for the organization of the paper.

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2. The actions of the Ptolemy-Thompson group $G_{mark} \cong T$ and the Kashaev group G_{dot} on tessellations

In this section, we study the certain infinite tessellations (triangulations) of the open unit disc $\mathbb{D}$, and two possible decorations on the tessellations: marked tessellations and dotted tessellations. We study the groups $G_{mark} \cong T$ and G_{dot} of changes of these enhanced tessellations, and study the natural relationship between them; the first one will be regarded as a 'universal' mapping class group. These groups are the two main ingredients of the present paper.

2.1. Tessellations of the unit disc.

This subsection provides a universal setting for the 'ideal triangulations' of hyperbolic surfaces. The surface we are going to deal with is the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ equipped with the usual hyperbolic (Poincaré) metric $ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}$, unless explicitly mentioned otherwise. We first need some definitions.

Definition 2.1. *An* ideal arc *(of the unit disc) connecting two distinct points on the unit circle $S^1 = \partial\mathbb{D}$ is a homotopy class of paths in $\mathbb{D}$ connecting the two points (one may want to do this in $\overline{\mathbb{D}}$). We don't distinguish the ideal arcs connecting the same set of endpoints (i.e. arcs have no orientation). The region bounded by three ideal arcs connecting three distinct points on S^1 is called an* ideal triangle.

In the figures appearing in the present paper (and usually in the hyperbolic geometry literature), ideal arcs are often assumed to be stretched to the unique hyperbolic geodesic (i.e. they are part of some circle, which intersects the unit circle at the right angle). We more or less follow [FuS] for basic definitions, in order to avoid the confusion as much as possible.

Definition 2.2. *A* tessellation τ *(of $\mathbb{D}$) is a (locally finite) triangulation of the unit disc $\mathbb{D}$ into ideal triangles, i.e. countable locally finite collection of ideal arcs whose complementary region*

is the disjoint union of (ideal) triangles. The vertices *of a tessellation are the endpoints of the ideal arcs constituting the tessellation. We denote by $\tau^{(0)}, \tau^{(1)}, \tau^{(2)}$ the collection of vertices of τ, the collection of ideal arcs (sometimes called* edges*) of τ, and the collection of ideal triangles formed by τ, respectively. The three ideal arcs bounding an ideal triangle are often called the* sides *of the triangle.*

The following definition will be very convenient when labeling the points of S^1 using $\mathbb{R}$:

Definition 2.3. *We denote by μ the* Cayley transform *from the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ to the unit disc $\mathbb{D}$, which also extends to their boundaries:*

$$\begin{array}{rrcl} & \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{RP}^1 & \longleftrightarrow & \overline{\mathbb{D}} = \mathbb{D} \cup S^1 \\ \mu : & x & \longmapsto & \mu(x) = \dfrac{x-i}{x+i}, \end{array} \tag{2.1}$$

where $\mathbb{RP}^1$ here is understood as $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$. When we say a Möbius transformation*, we mean an element of the automorphism group $PSL(2,\mathbb{R})$ of the hyperbolic space $\mathbb{H}$, often assumed to be extended to the boundary $\mathbb{RP}^1$ too, given by the fractional linear transformation*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in PSL(2,\mathbb{R}) : x \in \overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{RP}^1 \longmapsto \frac{ax+b}{cx+d} \in \overline{\mathbb{D}} = \mathbb{D} \cup S^1. \tag{2.2}$$

When we mention the action on $\overline{\mathbb{D}}$ of an element g of the group $PSL(2,\mathbb{R})$ or its subgroup $PSL(2,\mathbb{Z})$, we mean the conjugated action $\mu \circ g \circ \mu^{-1}$.

Via μ, each point on $S^1 = \partial\mathbb{D}$ gets labeled by the corresponding element of $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}$. If it's clear from the context, we will not bother to write μ all the time. For example, for $\alpha \in \mathbb{Q} \cup \{\infty\} \subset \mathbb{RP}^1$, the point $\mu(\alpha) \in S^1$ (often called a *rational point*)may be referred to as the 'point α on the circle' or even just α, especially in the pictures (e.g. see Fig. 1A). Since $\mathbb{Q} \cup \{\infty\}$ will come up often, we first settle the notation for its elements:

Definition 2.4. *A nonzero rational number is said to be in the* reduced expression *if it's written as $\frac{p}{q}$ with $p, q \in \mathbb{Z}, q > 0$, $gcd(p,q) = 1$. We set $\frac{0}{1}$ for the reduced expression for 0, and $\frac{1}{0}$ or $\frac{-1}{0}$ for the reduced expressions for ∞. So all rational numbers can be written uniquely in its reduced expression. We call the elements of $\mathbb{Q} \cup \{\infty\}$ the* extended rationals.

The most important example of tessellations is the 'Farey tessellation':

Definition 2.5. *The* Farey tessellation τ^* *is the tessellation whose vertices are (all the) rational points of S^1 (i.e. $\tau^{*(1)} = \mathbb{Q} \cup \{\infty\}$ via μ), and the two rational points $\mu(\frac{a}{b})$ and $\mu(\frac{c}{d})$ on S^1 (where $\frac{a}{b}$ and $\frac{c}{d}$ are reduced expressions; see Def. 2.4) are connected by an ideal arc of τ^* if and only if $|ad - bc| = 1$.*

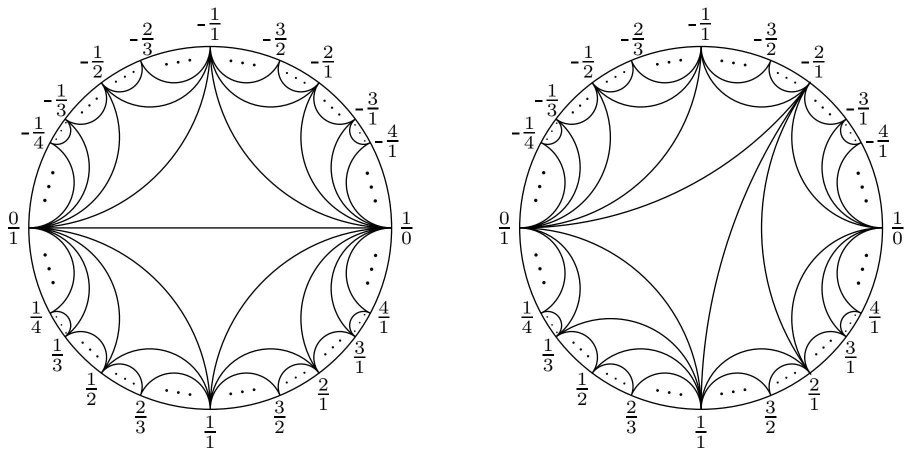
Alternatively, we can take the basic ideal triangle with the vertices $\mu(\frac{0}{1}), \mu(\frac{1}{0}), \mu(-\frac{1}{1}) \in S^1$, and the orbit of its sides under the $PSL(2,\mathbb{Z})$ action is the Farey tessellation. See Fig. 1A.

Remark 2.6. *Any ideal triangle of the Farey tessellation τ^* has the vertices $\mu(\frac{a}{b}), \mu(\frac{a+c}{b+d}), \mu(\frac{c}{d})$ for some extended rationals $\frac{a}{b}, \frac{c}{d}$ (in reduced expressions).*

A more general tessellation that we are interested in is as follows:

Definition 2.7. *A* Farey-type tessellation *is a tessellation whose vertices are (all the) rational points of S^1, all but finitely many of which ideal arcs are those of the Farey tessellation. See Fig. 1B. In the present paper, unless otherwise mentioned, a 'tessellation' would automatically mean a 'Farey-type tessellation'. Denote the set of all (Farey-type) tessellations by*

$$\mathcal{F}tess = \{\text{(Farey-type) tessellations } \tau\}. \tag{2.3}$$



(A) The Farey tessellation (B) A more general tessellation

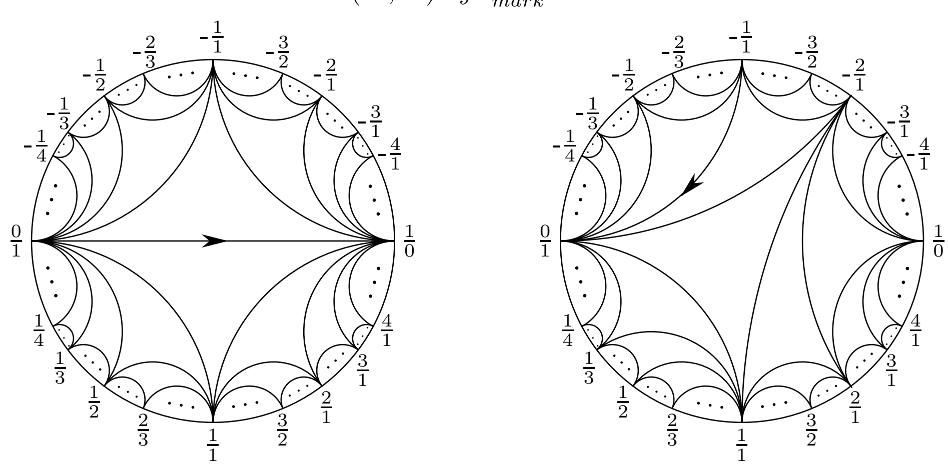
FIGURE 1. Examples of tessellations (μ is omitted in the vertex labels)

It is often necessary to put some decoration on the tessellation. One way is to specify the choice of an ideal arc with an orientation on it:

Definition 2.8. *A* marked tessellation $(\tau, \vec{a})$, *or a* tessellation with d.o.e. *(d.o.e. = distinguished oriented edge) is a tessellation τ with the choice of an oriented (ideal) arc $\vec{a}$, sometimes called the d.o.e. The d.o.e. $\vec{a}$ is indicated by an arrow in the picture (see Fig. 2). The* standard marked tessellation $(\tau^*, \vec{a}^*)$ *is the Farey tessellation τ^* with the d.o.e. $\vec{a}^*$ being the arc connecting $\mu(0)$ and $\mu(\infty)$ (with the direction $\mu(0) \to \mu(\infty)$); see Fig. 2A. We require that all but finitely many ideal arcs of a marked tessellation are those of the Farey tessellation τ^*. For a more general example, see Fig. 2B. Denote the set of all marked tessellations by*

$$\mathcal{F}tess_{mark} = \{\textit{marked (Farey-type) tessellations } (\tau, \vec{a})\}. \tag{2.4}$$

If the d.o.e. $\vec{a}$ is not ambiguous from the context, we will denote $(\tau, \vec{a})$ by τ_{mark} for convenience, and the standard marked tessellation $(\tau^*, \vec{a}^*)$ by τ^*_{mark}.



(A) The standard marked tessellation (B) A more general marked tessellation

FIGURE 2. Examples of marked tessellations

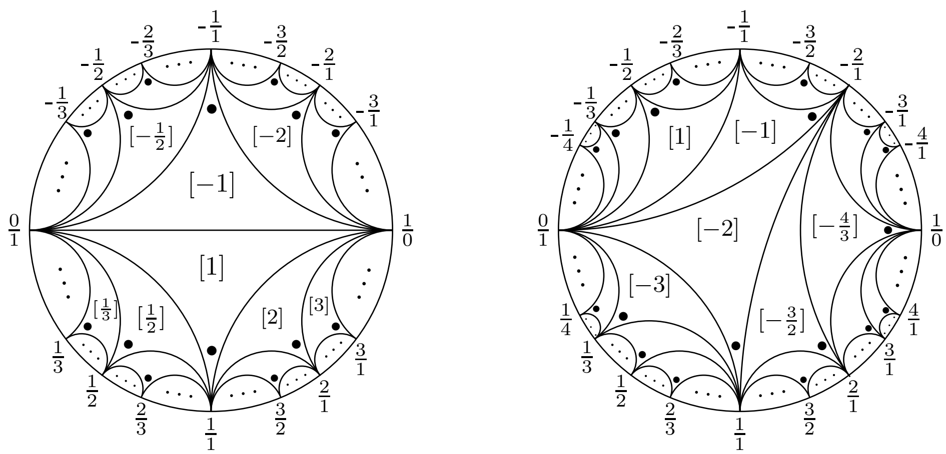
Another way of decorating a tessellation is to specify the choice of a corner in each ideal triangle, together with a labeling rule of the triangles:

Definition 2.9. *A* dotted tessellation (τ, D, L) *is a tessellation* τ *with a rule* D *which assigns to each triangle a distinguished corner, indicated by a dot* ($\bullet$) *in the picture (see Fig. 3), and a choice* L *of labeling of the triangles by* $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$*, i.e. a bijection between the set* $\tau^{(2)}$ *of ideal triangles of* τ *and* $\mathbb{Q}^\times$*, where in the picture we write* $[j]$ *for the triangle labeled by* $j \in \mathbb{Q}^\times$*; see Fig. 3 for examples.*

The standard dotted tessellation (τ^*, D^*, L^*) *is the Farey tessellation* τ^* *with the dots on the 'middle' vertices of the triangles (for a triangle with the vertices* $\mu(\frac{a}{b}), \mu(\frac{a+c}{b+d}), \mu(\frac{c}{d})$*, the 'middle vertex' is* $\mu(\frac{a+c}{b+d})$*; see Rem. 2.6), where the label of each triangle comes from the middle vertex; see Fig. 3A. We require that all but finitely many ideal triangles of a dotted tessellation to be those of the Farey tessellation* τ^* *with the choice of dots on them coinciding with that in the case for the standard dotted tessellation. For a more general example, see Fig. 3B. Denote the set of all dotted tessellations by*

$$\mathcal{F}tess_{dot} = \{\textit{dotted (Farey-type) tessellations } (\tau, D, L)\}. \tag{2.5}$$

If D *and* L *are not ambiguous from the context, we will denote* (τ, D, L) *by* τ_{dot} *for convenience, and the standard dotted tessellation* (τ^*, D^*, L^*) *by* τ^*_{dot}.



(A) The standard dotted tessellation (B) A more general dotted tessellation

FIGURE 3. Examples of dotted tessellations

Remark 2.10. *For the Farey tessellation* τ^* *(therefore for any Farey-type tessellation* τ *too), we saw that the set of vertices are naturally identified with the extended rationals* $\mathbb{Q} \cup \{\infty\}$*, and the set of triangles with the nonzero rationals* $\mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}$ *(by the 'middle vertex' of triangles). Each ideal arc except the one connecting* $\mu(0) = -1 \in S^1$ *and* $\mu(\infty) = 1 \in S^1$ *is contained in a unique ideal triangle in which the 'middle vertex' is to the opposite of the arc. If we label the arc by the rational number labeling that triangle (and the one connecting* $\mu(0)$ *and* $\mu(\infty)$ *by* -1*), then we get an identification of the set* $(\tau^*)^{(1)}$ *(hence* $\tau^{(1)}$*) of ideal arcs with* $\mathbb{Q} \setminus \{0, 1\}$.

There are natural maps

$$\mathcal{F}tess_{mark} \to \mathcal{F}tess \quad \text{and} \quad \mathcal{F}tess_{dot} \to \mathcal{F}tess \tag{2.6}$$

which forget the decorations of $\tau_{mark} = (\tau, \vec{a})$ and $\tau_{dot} = (\tau, D, L)$ and returns just the underlying tessellation τ. Let us now introduce a mapping from $\mathcal{F}tess_{mark}$ to $\mathcal{F}tess_{dot}$, mimicking the relationship between the standard objects $\tau^*_{mark} = (\tau^*, \vec{a}^*)$ and $\tau^*_{dot} = (\tau^*, D^*, L^*)$:

Definition 2.11. *Define the mapping*

$$F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot} \tag{2.7}$$

as follows. We require that this map does not change the underlying tessellation. Given a marked tessellation $(\tau, \vec{a}) \in \mathcal{F}tess_{mark}$, we should specify how we assign dots to each triangle in $\tau^{(2)}$, and how to give $\mathbb{Q}^\times$-labels to the triangles. To do this, we first build the extended rational number τ_j for every extended rational number j by the following 'inductive' process:

1. *Let τ_0 and τ_∞ be the two extended rational numbers labeling (via μ) the starting point and the ending point of the d.o.e. $\vec{a}$, respectively.*
2. *Among the ideal triangles in $\tau^{(2)}$, there are two ideal triangles having $\vec{a}$ as one of their sides. Take the one which is 'located at the right (resp. left) of $\vec{a}$', and let τ_1 (resp. τ_{-1}) be the extended rational number labeling the third vertex of this triangle. Fig. 4B shows the assignment of $\tau_0, \tau_\infty, \tau_1, \tau_{-1}$ to the marked tessellation $(\tau, \vec{a})$ in Fig. 2B (in that particular example, we have $\tau_0 = -1, \tau_\infty = 0, \tau_1 = -\frac{1}{2}, \tau_{-1} = -2$).*
3. *If we have an ideal triangle in $\tau^{(2)}$ having two vertices whose extended-rational-number labels are already identified by this process as $\tau_{\frac{a}{b}}$ and $\tau_{\frac{c}{d}}$ for some extended rational numbers $\frac{a}{b}, \frac{c}{d}$ (where these are in the reduced expressions), then we let $\tau_{\frac{a+c}{b+d}}$ be the extended rational number labeling the third vertex of this triangle. 'Repeat' this step.*

Analogously to the case of the Farey tessellation (see Rem. 2.6), we now can see that the vertices of any ideal triangle in $\tau^{(2)}$ are labeled (via μ) by the rational numbers $\tau_{\frac{a}{b}}$, $\tau_{\frac{a+c}{b+d}}$, $\tau_{\frac{c}{d}}$ for some extended rationals $\frac{a}{b}, \frac{c}{d}$ (in the reduced expressions). We choose the corner labeled by $\tau_{\frac{a+c}{b+d}}$ as the distinguished corner of this triangle (i.e. put the dot in that corner, for this triangle); this is the 'dotting rule D'. And we label this triangle by $\frac{a+c}{b+d}$; this is the 'labeling rule L for triangles'.

This process of choosing the distinguished corners and the $\mathbb{Q}^\times$-labels for the triangles in $\tau^{(2)}$ described above is well-defined. We now define $F((\tau, \vec{a}))$ to be this resulting dotted-tessellation (τ, D, L).

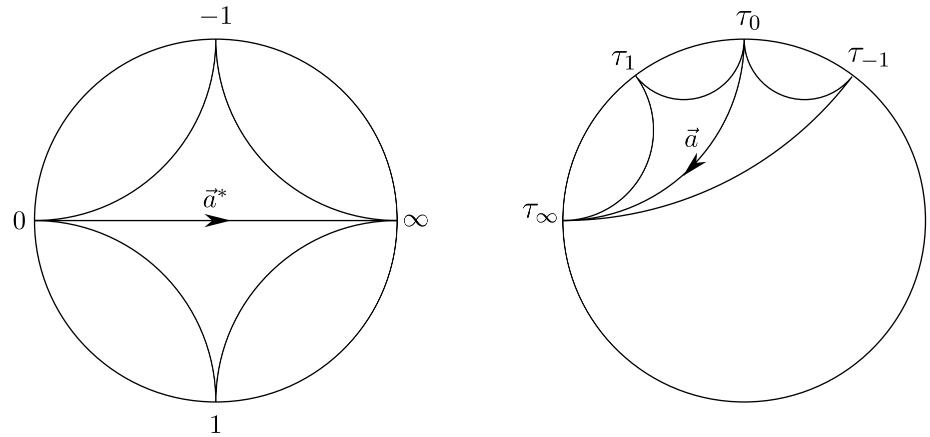
For example, the image of the standard marked tessellation τ^*_{mark} (Fig. 2A) under F is the standard dotted tessellation τ^*_{dot} (Fig. 3A), and the image of the marked tessellation in Fig. 2B under F is the dotted tessellation in Fig. 3B. One can easily see the following.

Proposition 2.12. *The map F in* (2.7) *is injective, and $\mathcal{F}tess_{dot}$ is strictly larger than $F(\mathcal{F}tess_{mark})$.*

One may wonder how natural the choice of this map $F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot}$ is. To see this, we first need the following definition.

Definition 2.13. *A homeomorphism of the open unit disc $\mathbb{D}$ to itself is said to be* asymptotically rigid *if its continuous extension to the boundary circle $S^1 = \partial\mathbb{D}$ is a piecewise-$PSL(2,\mathbb{Z})$ homeomorphism of S^1 with finite number of pieces, with the breaking points for the pieces being rational points on S^1. The group of such piecewise-$PSL(2,\mathbb{Z})$ homeomorphisms of S^1 is denoted by $PPSL(2,\mathbb{Z})$. An asymptotically rigid homeomorphism of $\mathbb{D}$ is called* globally rigid *if its extension to S^1 is a $PSL(2,\mathbb{Z})$ homeomorphism.*

One can observe that the asymptotically rigid homeomorphism of $\mathbb{D}$ form a group, and each of them permutes the rational points on the boundary S^1 (by the continuous extension to S^1).



(A) Four basic vertices of the standard marked tessellation $\tau^*_{mark} = (\tau^*, \vec{a}^*)$

(B) Four basic vertices of a more general marked tessellation $(\tau, \vec{a})$

FIGURE 4. Four basic vertices of marked tessellations

Definition 2.14. *Two dotted tessellations τ_{dot} and τ'_{dot} are said to be* in the same (dot-)configuration *if one can be obtained by applying to the other an isotopy class of asymptotically rigid homeomorphisms of $\mathbb{D}$. Roughly, we can think of* a configuration (of triangle labels and dots) *as an equivalence class of the labeling rule and dotting rule with respect to the above defined relation (being in a same configuration).*

Now, the following (easily seen) observation somewhat justifies the choice of F.

Proposition 2.15. *All the images of F in $\mathcal{F}tess_{dot}$ are in the same configuration as the standard dotted tessellation τ^*_{dot}, and any dotted tessellation in the same configuration as τ^*_{dot} is realized as an image of F.*

In the following subsection, we shall observe that the isotopy classes of asymptotically rigid homeomorphisms of $\mathbb{D}$ act on $\mathcal{F}tess_{mark}$, as well as on $\mathcal{F}tess_{dot}$, and that the action on $\mathcal{F}tess_{mark}$ is transitive. From the definition of F (Def. 2.11), it is not difficult to see the following, which immediately yields Prop. 2.15:

Proposition 2.16. *The map $F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot}$ as defined in Def. 2.11 is equivariant under the action of the isotopy classes of asymptotically rigid homomorphisms of $\mathbb{D}$ (Def. 2.13).*

Moreover, the condition $F(\tau^*_{mark}) = \tau^*_{dot}$ and this equivariance completely determines $F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot}$, recovering Def. 2.11.

2.2. The actions of G_{mark} and G_{dot} on the enhanced tessellations $\mathcal{F}tess_{mark}$ and $\mathcal{F}tess_{dot}$.

In this subsection we study some automorphisms of $\mathcal{F}tess_{mark}$ and $\mathcal{F}tess_{dot}$. We first look at the action on the marked tessellations.

Definition 2.17. *Let G_{mark} be the group of automorphisms of $\mathcal{F}tess_{mark}$ generated by the automorphisms α and β:*

$$G_{mark} = \langle \alpha, \beta \rangle, \tag{2.8}$$

where α and β are defined as in Figures 5 and 6, leaving the other part (denoted by triple dots '$\cdots$') intact. More precisely,

(1) *For the α action on $(\tau, \vec{a}) \in \mathcal{F}tess_{mark}$, first locate the two triangles in $\tau^{(2)}$ having the d.o.e. $\vec{a}$ as one of their sides (see Def. 2.11). These two triangles form an ideal*

quadrilateral; replace the d.o.e. with the other (ideal) diagonal of this quadrilateral, with the orientation given as if the d.o.e. is rotated counterclockwise, and leave all other ideal arcs in τ intact. See Fig. 5.

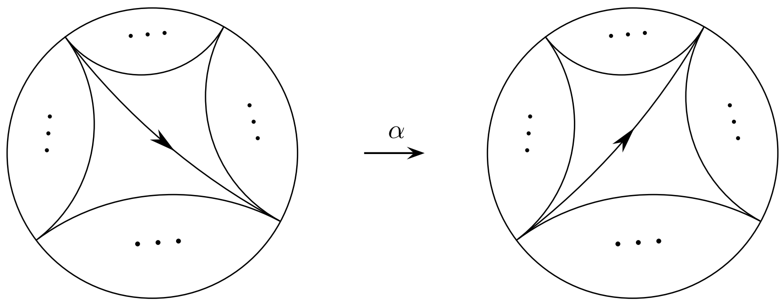


FIGURE 5. The action of α on $\mathcal{F}tess_{mark}$

(2) *For the β action on $(\tau, \vec{a})$, first locate the three vertices which are labeled by the extended-rational-number labels $\tau_0, \tau_\infty, \tau_{-1}$ as described in Def. 2.11. In particular, the d.o.e. $\vec{a}$ runs from τ_0 to τ_∞. We don't change the tessellation τ, but we denote the underlying tessellation of the new marked tessellation by τ', so that $\tau = \tau'$. Let the arc in $(\tau')^{(1)}$ connecting τ'_∞ and τ'_{-1} to be the new d.o.e. $\vec{a}'$, with the direction $\tau'_\infty \to \tau'_{-1}$. We now let $\beta((\tau, \vec{a})) = (\tau', \vec{a}')$. See Fig. 6.*

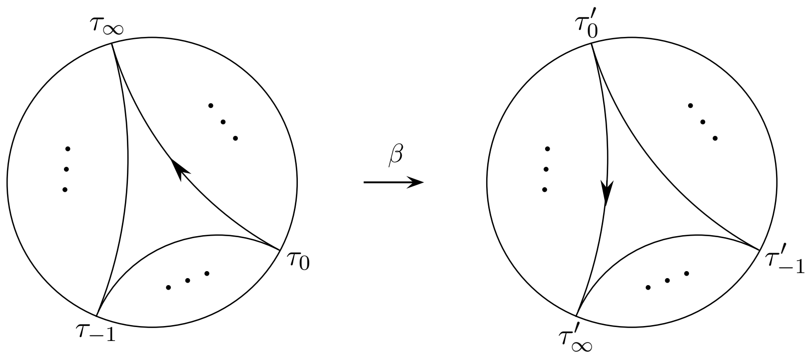


FIGURE 6. The action of β on $\mathcal{F}tess_{mark}$

Proposition 2.18. *The group G_{mark} acts transitively on $\mathcal{F}tess_{mark}$.*

Proof. Recall that the α action replaces a diagonal of some ideal quadrilateral with the other diagonal. If we forget the marking and just think of the underlying tessellations, we can call this change of tessellations a 'flip' (this will appear again in Def. 2.25). We can associate the

flip to any ideal arc of a tessellation. It is easy to see that any two (Farey-type) tessellations are related by a finite number of flips. Given any underlying tessellation, we can move the d.o.e. to any ideal arc with any orientation while fixing the underlying tessellation, using a finite number of β's and α^2's. Since α induces the flip of the underlying Farey-type tessellation along the d.o.e., and we know how to change the d.o.e. to any ideal arc in a given underlying tessellation by a finite number of moves, we conclude that any two marked tessellations are related by the composition of a finite number of α's and β's. □

We shall also prove that G_{mark} acts freely on $\mathcal{F}tess_{mark}$. Before doing so, we first notice that that the actions α and β are induced by asymptotically rigid homeomorphisms of $\mathbb{D}$ to itself (see Def. 2.13). More precisely, we can observe:

Proposition 2.19. *The action of an element g of the group G_{mark} on the standard marked tessellation τ^*_{mark} is induced by an isotopy class of asymptotically rigid homeomorphisms φ_g of $\mathbb{D}$ (see Def. 2.13). Any marked tessellation τ_{mark} can be written as $h.\tau^*_{mark}$ for some $h \in G_{mark}$ (Prop. 2.18) and hence as $\varphi_h.\tau^*_{mark}$. The the action of the homeomophism class $\varphi_h\varphi_g\varphi_h^{-1}$ of $\mathbb{D}$ on $\tau_{mark} = \varphi_h.\tau^*_{mark}$ induces the action of g on τ_{mark}. In fact, G_{mark} is the group of* all *automorphisms of $\mathcal{F}tess_{mark}$ induced from asymptotically rigid homeomorphisms of $\mathbb{D}$ as above.*

Hence, G_{mark} can be viewed as an 'asymptotically rigid' mapping class group of $\mathbb{D}$ (together with rational boundary points), which can be thought of as a universal mapping class group. Since any element of G_{mark} is completely determined by the action on the rational boundary points of the homeomorphism of $\mathbb{D}$ corresponding to its action on any fixed single marked tessellation, G_{mark} can also be identified with $PPSL(\mathbb{Z})$ (Def. 2.13).

It is not too difficult to show that any automorphism of $\mathcal{F}tess_{mark}$ induced from an asymptotically rigid homeomorphism of $\mathbb{D}$ can be expressed as the finite composition of α's and β's. See [P2] for a discussion of the universal mapping class group.

Now we can prove that the action of G_{mark} on $\mathcal{F}tess_{mark}$ is free:

Proposition 2.20. *The group G_{mark} acts freely on $\mathcal{F}tess_{mark}$.*

Proof. Suppose $g \in G_{mark}$ fixes τ^*_{mark}. From Prop. 2.19 there exist an asymptotically rigid homeomorphism class $\varphi_g \in G_{mark}$ whose action on τ^*_{mark} induces the g-action on τ^*_{mark}, i.e. $g.\tau^*_{mark} = \varphi_g.\tau^*_{mark}$; here we have $g.\tau^*_{mark} = \tau^*_{mark}$. As in Prop. 2.19, any $\tau_{mark} \in \mathcal{F}tess_{mark}$ can be written as $\tau_{mark} = \varphi_h.\tau^*_{mark}$ for some homeomorphism class, and the result of the action of g on it is given by $(\varphi_h\varphi_g\varphi_h^{-1}).(\varphi_h.\tau^*_{mark}) = \varphi_h.(\varphi_g.\tau^*_{mark}) = \varphi_h.\tau^*_{mark} = \tau_{mark}$, hence g also fixes τ_{mark}. Therefore g is the identity automorphism of $\mathcal{F}tess_{mark}$. □

This group G_{mark} can also be finitely presented as follows.

Theorem 2.21 (Lochak-Schneps [LoSc]; see also Prop. 5.7 of the present paper)**.** *The group G_{mark} has a following presentation by generators and relations:*

$$G_{mark} = \left\langle \alpha, \beta \,\middle|\, \begin{array}{l} (\beta\alpha)^5 = \alpha^4 = \beta^3 = 1, \\ \big[\beta\alpha\beta,\, \alpha^2\beta\alpha\beta\alpha^2\big] = \big[\beta\alpha\beta,\, \alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2\big] = 1 \end{array} \right\rangle, \tag{2.9}$$

so G_{mark} is isomorphic to the Ptolemy-Thompson group T *appearing in* [FuS].

We will use the notations G_{mark} and T interchangeably in the present paper; see Prop. 5.7 for justification. The equations $\alpha^4 = 1$ and $\beta^3 = 1$ are manifest from the picture. The famous *pentagon equation* $(\beta\alpha)^5 = 1$ is also easily checked by drawing the corresponding pictures.

Remark 2.22. *As pointed out in Remark 2.3 of* [FuKap2]*, the subgroup of $G_{mark} \cong T \cong PPSL(2,\mathbb{Z})$ generated by α^2 and β is isomorphic to $PSL(2,\mathbb{Z})$. As in Prop. 2.19 of the present paper, $PSL(2,\mathbb{Z})$ is the subgroup of all elements of G_{mark} induced by the 'globally rigid' homeomorphisms of $\mathbb{D}$ (see Def. 2.13).*

Now we proceed to study the actions on dotted tessellations. However, unlike the case for marked tessellations, it is often considered more natural to consider a groupoid, instead of a group.

Definition 2.23. *Let the* Ptolemy groupoid *Pt be the category whose objects are the (Farey-type) tessellations $\tau \in \mathcal{F}tess$, and for any two objects τ, τ', there is exactly one morphism denoted by $[\tau, \tau']$. We set the composition of morphisms by*

$$[\tau', \tau''] \circ [\tau, \tau'] = [\tau, \tau''] \tag{2.10}$$

just like the composition of functions. Analogously, define the dotted Ptolemy groupoid *Pt_{dot} to be the category whose objects are dotted tessellations $\tau_{dot} \in \mathcal{F}tess_{dot}$, and for any two objects τ_{dot}, τ'_{dot} there is exactly one morphism denoted by $[\tau_{dot}, \tau'_{dot}]$.*

Remark 2.24. *Some authors use the composition rule written in an opposite order to* (2.10)*. Then the covariant formalism in §3.1 has to be replaced by the contravariant formalism.*

Each morphism $[\tau, \tau']$ of Pt can be thought of as a 'change of triangulation' from τ to τ'. Among them, there are elementary ones, described as follows.

Definition 2.25. *For any edge e of a (Farey-type) tessellation τ, there are exactly two ideal triangles $\in \tau^{(2)}$ having e as one of their sides. These two triangles form an ideal quadrilateral, with e as a diagonal arc. Replace e with the other diagonal arc of this quadrilateral, to obtain the new tessellation τ'. This $[\tau, \tau']$ is called the* flip of τ with respect to e*.*

One can easily observe:

Proposition 2.26. *Any two (Farey-type) tessellations τ, τ' are related by a finite number of flips.*

These flips, viewed as morphisms of Pt, satisfy some algebraic relations by the requirement that there is only one morphism from an object of Pt to another. For example, if we flip the same edge twice, we arrive at the same tessellation, hence this says the "twice-flip is the identity". Also, there is another important relation involving five flips (called the *pentagon relation*), but these are not dealt with now, because they will follow later from the study of morphisms of Pt_{dot}.

The 'flips', as defined in Def. 2.25, requires the choice of a tessellation τ and one of its arcs $e \in \tau^{(1)}$. Unlike this, we'll give names to the three types of elementary morphisms of Pt_{dot} that we will describe in Def. 2.27 in terms only of the labels of the involved triangles, by making use of the fixed label index set $\mathbb{Q}^\times$ for the triangles.

Definition 2.27. *We describe the* elementary moves *$A_{[j]}$, $T_{[j][k]}$, $P_{(jk)}$ of Pt_{dot} for $j, k \in \mathbb{Q}^\times$ (triangle labels, where $j \neq k$), each of which represents some class of morphisms of Pt_{dot}.*

1) *If $\tau'_{dot} \in \mathcal{F}tess_{dot}$ is obtained from $\tau_{dot} \in \mathcal{F}tess_{dot}$ by moving the dot $\bullet$ (i.e. the distinguished corner) of the triangle of τ labeled by $j \in \mathbb{Q}^\times$ counterclockwise to the next corner in that triangle, while leaving all other information intact, then we name the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} by $A_{[j]}$; see Fig. 8. There are infinitely many morphisms of Pt_{dot} which are named by $A_{[j]}$ by this rule.*

2) *Suppose that for $\tau_{dot} \in \mathcal{F}tess_{dot}$, the triangles of τ labeled by j and k (where $j \neq k$) are adjacent to each other (i.e. share one side) and that the dots of those two triangles are exactly as in the LHS of Fig. 7 (relative to the common arc of the two triangles). If τ'_{dot} is obtained from τ_{dot} by replacing the common arc of the triangles labeled by j, k by the other diagonal arc of the ideal quadrilateral formed by those two triangles, and setting the new dots and labels as in the RHS of Fig. 7, as if we rotate* clockwise *the diagonal arc of the quadrilateral while letting the dots $\bullet$ and triangle labels $[j], [k]$ be 'floating' and thus pushed accordingly by the rotating arc, while leaving all the other information intact, then we name the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} by $T_{[j][k]}$; see Fig. 7.*
3) *If τ'_{dot} is obtained from τ_{dot} by exchanging the labels of the two triangles labeled by $j, k \in \mathbb{Q}^\times$ and leaving all the other information intact, then we name the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} by $P_{(jk)}$. We sometimes call $P_{(jk)}$ an* index permutation. *In the same spirit, for any permutation γ of $\mathbb{Q}^\times$, we denote by P_γ the corresponding index permutation, which relabels each triangle by $j \mapsto \gamma(j)$.*

In each of the above cases, we say τ'_{dot} is obtained from τ_{dot} by applying *the relevant move. Any morphism of Pt_{dot} corresponding to one of the cases above is called an* elementary morphism *of Pt_{dot}.*

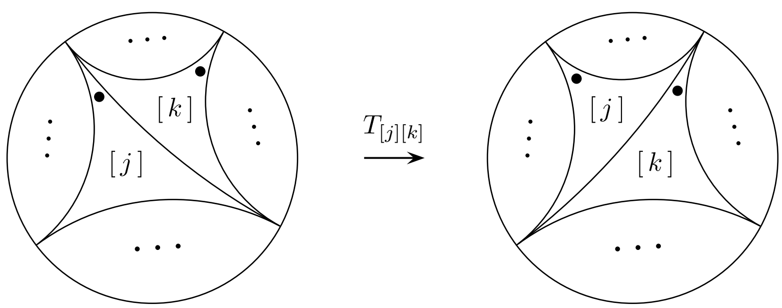


FIGURE 7. The action of $T_{[j][k]}$ on $\mathcal{F}tess_{dot}$

So, any elementary morphism of Pt_{dot} is represented by an elementary move. It's easy to see the following:

Proposition 2.28. *The morphism between any two objects of Pt_{dot} (i.e. two elements of $\mathcal{F}tess_{dot}$) can be written as the composition of a finite number of elementary morphisms, hence can be represented as the composition of a finite number of elementary moves.*

As usual, we read the composition of (or 'a word in') the elementary moves from the right; for example, $A_{[j]}T_{[j][k]}$ means applying $T_{[j][k]}$ first and then $A_{[j]}$. The elementary move $T_{[j][k]}$ is the enhanced version of the 'flip' for Pt defined in Def. 2.25. The elementary moves of Pt_{dot} satisfy some algebraic relations, for example the one corresponding to the previously mentioned "twice-flip is the identity", and the pentagon relation for the 'flips' as mentioned briefly before.

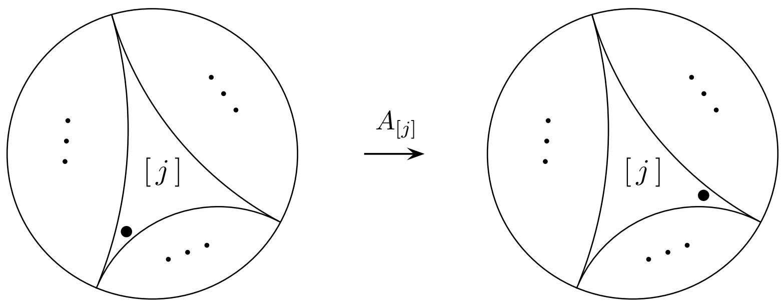


FIGURE 8. The action of $A_{[j]}$ on $\mathcal{F}tess_{dot}$

Theorem 2.29 (See Teschner [Te] for proof; see also Kashaev [Kas3]). *All the nontrivial algebraic relations among the elementary moves of Pt_{dot} are the consequences of*

$$A_{[j]}^3 = id, \tag{2.11}$$

$$T_{[k][\ell]}T_{[j][k]} = T_{[j][k]}T_{[j][\ell]}T_{[k][\ell]}, \tag{2.12}$$

$$A_{[j]}T_{[j][k]}A_{[k]} = A_{[k]}T_{[k][j]}A_{[j]}, \tag{2.13}$$

$$T_{[j][k]}A_{[j]}T_{[k][j]} = A_{[j]}A_{[k]}P_{(jk)}, \tag{2.14}$$

where $j, k, \ell \in \mathbb{Q}^\times$ are mutually distinct. Also there are trivial relations, satisfied by the index permutations $P_{(jk)}$

$$P_{(jk)}^2 = id, \quad P_{(jk)} f_{\cdots,j,\cdots,k,\cdots} P_{(jk)} = f_{\cdots,k,\cdots,j,\cdots}, \quad P_{(jk)} = P_{(kj)}, \tag{2.15}$$

where $f_{\cdots,j,\cdots,k,\cdots}$ is any composition of the elementary moves (conjugation by $P_{(jk)}$ results in exchanging the subscripts j and k), and that any two words (i.e. composition) in the elementary moves whose collections of subscripts (indices) don't intersect with each other commute (for example, $A_{[j]}T_{[j][k]}$ and $A_{[\ell]}$ commute if j, k, ℓ are mutually distinct).

Each of the above relations is meant such that whenever the LHS can be applied to some $\tau_{dot} \in \mathcal{F}tess_{dot}$, then the RHS can also be applied to τ_{dot} and they yield the same result τ'_{dot}.

We find it convenient to define an abstract group with generators and relations, using Def. 2.27 and Thm. 2.29.

Definition 2.30 (See Frenkel-Kim [FrKi]). *For any index set I, define the* Kashaev group G_I associated to I *by generators and relations, with the generators $A_{[j]}, T_{[j][k]}, P_{(jk)}$ ($j, k \in I$, $j \neq k$) and the relations* (2.11), (2.12), (2.13), (2.14),(2.15), *and the commuting relation mentioned at the end of Thm. 2.29.*

For our case when $I = \{\text{triangle lables}\} = \mathbb{Q}^\times$, we denote this group by

$$G_{dot} = G_{\mathbb{Q}^\times} = \langle A_{[j]}, T_{[j][k]}, P_{(jk)} \rangle / (\text{relations mentioned in Thm. 2.29}), \tag{2.16}$$

which can be thought of as the formal group of changes of dotted tessellations.

Remark 2.31. *To be more precise, we should let the group G_{dot} also include the more general index permutations P_γ (for permutations γ of $\mathbb{Q}^\times$; see Def. 2.27 for P_γ); see* (2.26).

Remark 2.32 (see e.g. [FuS]). *We can define the 'marked Ptolemy groupoid Pt_{mark}' analogously as in Def. 2.23. Penner* [P2] *constructed a group out of the groupoid Pt_{mark}, by the similar argument as we used to obtain G_{dot} out of Pt_{dot} from Definitions 2.27 and 2.30. He called the resulting group the* (univeral) Ptolemy group, *which was found to be isomorphic to the* Thompson group T *of dyadic piecewise affine homeomorphisms of S^1. This justifies the name* Ptolemy-Thompson group *for this group.*

By Thm. 2.29, the 'action' of G_{dot} on $\mathcal{F}tess_{dot}$ is 'free', in the following sense:

Corollary 2.33. *If $g \in G_{dot}$ fixes one element of $\mathcal{F}tess_{dot}$, i.e. $g.\tau_{dot} = \tau_{dot}$ for some $\tau_{dot} \in \mathcal{F}tess_{dot}$, then $g = 1$. It's then easy to see also that if $g, g' \in G_{dot}$ are applicable to some τ_{dot} and if $g.\tau_{dot} = g'.\tau_{dot}$ holds, then $g = g'$. Therefore any element of G_{dot} which can be applied to at least one dotted tessellation is completely characterized by its action on one element of $\mathcal{F}tess_{dot}$ (to which it can be applied).*

2.3. **The natural map from G_{mark} to G_{dot}.** Now that G_{dot} will play a role of the group of automorphisms on $\mathcal{F}tess_{dot}$, we'd like to establish a relationship between G_{dot} and the automorphism group G_{mark} of $\mathcal{F}tess_{mark}$, in the following sense. Using the injective map (2.7) $F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot}$, we wish to establish a natural group homomorphism

$$\mathbf{F} : G_{mark} \to G_{dot} \tag{2.17}$$

making the following diagram to commute for any $g \in G_{mark}$:

$$\begin{tikzcd} \mathcal{F}tess_{mark} \arrow[r, "F"] \arrow[d, "g"] & \mathcal{F}tess_{dot} \arrow[d, "\mathbf{F}g"] \\ \mathcal{F}tess_{mark} \arrow[r, "F"] & \mathcal{F}tess_{dot}, \end{tikzcd} \tag{2.18}$$

that is,

$$F(g.\tau_{mark}) = (\mathbf{F}g).(F(\tau_{mark})), \quad \forall \tau_{mark} \in \mathcal{F}tess_{mark}, \quad \forall g \in G_{mark}. \tag{2.19}$$

It is a priori not obvious that such a map $\mathbf{F}$ satisfying (2.19) exists. Fix $g \in G_{mark}$. For any given $\tau_{mark} \in \mathcal{F}tess_{mark}$, we get two elements $F(\tau_{mark})$ and $F(g.\tau_{mark})$ of $\mathcal{F}tess_{dot}$, hence a unique morphism $[F(\tau_{mark}), F(g.\tau_{mark})]$ in Pt_{dot} (Def. 2.23). This morphism can be represented as a unique element of G_{dot}, by Prop. 2.28 and Cor. 2.33. However, we should check that a different choice of $\tau_{mark} \in \mathcal{F}tess_{mark}$ leads to the same element of G_{dot}; only then, we have a well-defined image $\mathbf{F}g \in G_{dot}$ of $g \in G_{mark}$.

Pick any $\tau_{mark} \in \mathcal{F}tess_{mark}$. From Prop. 2.18, there is $h \in G_{mark}$ such that $h.\tau^*_{mark} = \tau_{mark}$. By Prop. 2.19, there are (isotopy classes of) asymptotically rigid homeomorphisms φ_h and φ_g of $\mathbb{D}$ to itself, inducing respectively the actions of h and g on τ^*_{mark}, that is, we can write $\varphi_h.\tau^*_{mark} = \tau_{mark}$ and $\varphi_g.\tau^*_{mark} = \tau_{mark}$. Furthermore, Prop. 2.19 says that the homeomorphism $\varphi_h \varphi_g \varphi_h^{-1}$ of $\mathbb{D}$ induces the action of $g \in G_{mark}$ on $\tau_{mark} = \varphi_h.\tau^*_{mark}$. Therefore we can see that

$$g.\tau_{mark} = (\varphi_h \varphi_g \varphi_h^{-1}).(\varphi_h.\tau^*_{mark}) = \varphi_h.(\varphi_g.\tau^*_{mark}) = \varphi_h.(g.\tau^*_{mark}). \tag{2.20}$$

Now, we can observe that the morphism

(2.21)

$$[F(\tau_{mark}),\, F(g.\tau_{mark})] = [F(\varphi_h.\tau^*_{mark}),\, F(\varphi_h.(g.\tau^*_{mark})] \overset{Prop. 2.16}{=} [\varphi_h.(F(\tau^*_{mark})),\, \varphi_h.(F(g.\tau^*_{mark}))]$$

of Pt_{dot} indeed is represented by the same element of G_{dot} as the morphism $[F(\tau^*_{mark})),\ F(g.\tau^*_{mark})]$ (it's not hard to see this), and therefore $g \in G_{mark}$ induces a well-defined element of G_{dot}, and we let that element be $\mathbf{F}g$.

We will first figure out what elements of G_{dot} the images $\mathbf{F}\alpha$ and $\mathbf{F}\beta$ of α and β have to be, in order for them to satisfy (2.19). Since F is injective, it is a bijection from $\mathcal{F}tess_{mark}$ to its image $F(\mathcal{F}tess_{mark}) \subset \mathcal{F}tess_{dot}$. Hence, in fact,

$$\mathbf{F}\alpha = F \circ \alpha \circ F^{-1} \quad \text{and} \quad \mathbf{F}\beta = F \circ \beta \circ F^{-1} \tag{2.22}$$

hold, both sides of both equations acting on $F(\mathcal{F}tess_{mark})$. Now, we extend $\mathbf{F}$ to the group G_{dot} by requiring it to be a group homomorphism (i.e. preserves the group multiplications); so, on any word in α, β we define

$$\mathbf{F}(\alpha^{n_1}\beta^{m_1}\alpha^{n_2}\cdots\alpha^{n_r}\beta^{m_r}) = \mathbf{F}(\alpha)^{n_1}\mathbf{F}(\beta)^{m_1}\cdots\mathbf{F}(\alpha)^{n_r}\mathbf{F}(\beta)^{m_r}, \tag{2.23}$$

for $r \geq 1$, where n_j, m_j are any integers. For this definition of $\mathbf{F}$ on G_{mark} to be well-defined, for any word in α, β which equals $1 \in G_{mark}$ (by the relations of α, β, e.g. $(\beta\alpha)^5 = 1$), the element of G_{dot} obtained from this word by replacing α, β with $\mathbf{F}\alpha, \mathbf{F}\beta$ should equal $1 \in G_{dot}$. This is indeed true, because from (2.22) we have $\mathbf{F}g = F \circ g \circ F^{-1}$ for any $g \in G_{mark}$.

Since the G_{dot}-action on $\mathcal{F}tess_{dot}$ is 'free' (Cor. 2.33), we now only need to figure out what $(\mathbf{F}\alpha).\tau^*_{dot}$ and $(\mathbf{F}\beta).\tau^*_{dot}$ are, where τ^*_{dot} is the standard dotted tessellation. So we can just put $\tau_{mark} = \tau^*_{mark}$ (the standard marked tessellation; see Def. 2.8) into (2.19) and compute the LHS, since $F(\tau^*_{mark}) = \tau^*_{dot}$ (see the remarks following Def. 2.11). To write this result, we first need the following definition.

Definition 2.34. *Let P_{γ_α}, P_{γ_β} be the index permutations corresponding to the permutations $\gamma_\alpha, \gamma_\beta$ of $\mathbb{Q}^\times$ induced by the triangle label changes which occur when applying α, β, respectively, described as follows.*

We first require that γ_α fixes -1 and 1. The set of ideal triangles of τ^*_{mark} (where each ideal triangle is viewed in this definition just as a subset of $\mathbb{D}$) is same as that of $\alpha.\tau^*_{mark}$ except those labeled by -1 and 1 (we use the triangle labeling rules L obtained by applying F to the marked tessellations). If an ideal triangle is labeled by j ($j \neq -1, 1$) under the labeling rule L^* of $F(\tau^*_{mark}) = \tau^*_{dot} = (\tau^*, D^*, L^*)$ and is labeled by j' of the labeling rule of $F(\alpha.\tau^*_{mark})$, then we set $\gamma_\alpha(j) = j'$. We thus establish a $\mathbb{Q}^\times$-permutation γ_α.

Defining γ_β is similar but easier, since β doesn't change the underlying tessellation. If an ideal triangle is labeled by $j \in \mathbb{Q}^\times$ by the labeling rule of $F(\tau^*_{mark})$ and is labeled by j' by the labeling rule of $F(\beta.\tau^*_{mark})$, then we set $\gamma_\beta(j) = j'$. We thus get a $\mathbb{Q}^\times$-permutation γ_β, and it follows that $\gamma_\beta(-1) = -1$.

The permutations γ_α and γ_β are best seen in the pictures; see Figures 9 and 10. We can write some of the actions of the permutations γ_α, γ_β on $\mathbb{Q}^\times$:

$$\begin{cases} \gamma_\alpha(-1) = -1, \quad \gamma_\alpha(1) = 1, \quad \gamma_\alpha(-2) = 2, \quad \gamma_\alpha(-\frac{1}{2}) = -2, \\ \gamma_\alpha(\frac{1}{2}) = -\frac{1}{2}, \quad \gamma_\alpha(2) = \frac{1}{2}, \quad \gamma_\alpha(\frac{1}{3}) = -\frac{3}{2}, \quad \gamma_\alpha(3) = \frac{2}{3}, \quad \text{etc,} \end{cases} \tag{2.24}$$

$$\begin{cases} \gamma_\beta(-1) = -1, \quad \gamma_\beta(1) = -\frac{1}{2}, \quad \gamma_\beta(-2) = 1, \quad \gamma_\beta(-\frac{1}{2}) = -2, \\ \gamma_\beta(\frac{1}{2}) = -\frac{2}{3}, \quad \gamma_\beta(2) = -\frac{1}{3}, \quad \gamma_\beta(\frac{1}{3}) = -\frac{3}{4}, \quad \gamma_\beta(3) = -\frac{1}{4}, \quad \text{etc.} \end{cases} \tag{2.25}$$

The formulas for $\mathbf{F}\alpha$ and $\mathbf{F}\beta$ can be easily obtained by looking at Figures 9 and 10. We write this result as:

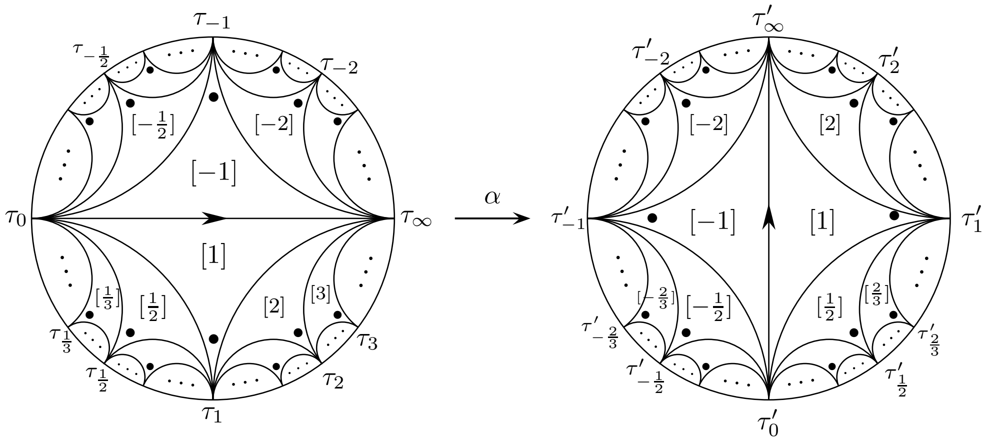


FIGURE 9. The triangle label change P_{γ_α}, associated to α

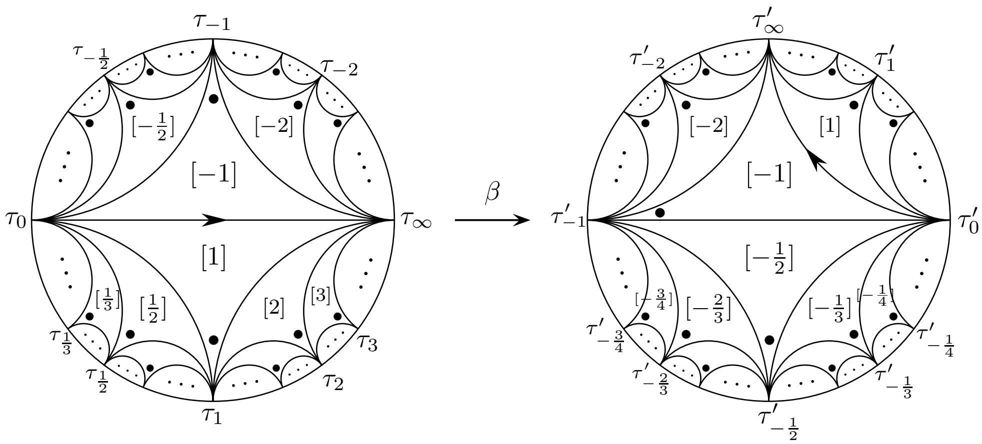


FIGURE 10. The triangle label change P_{γ_β}, associated to β

Proposition 2.35. *There exists a unique group homomorphism* $\mathbf{F} : G_{mark} \to G_{dot}$ *which satisfies* (2.19)*, and it is given by*

$$\mathbf{F}\alpha = A_{[-1]} T^{-1}_{[-1][1]} A_{[1]} P_{\gamma_\alpha}, \qquad \mathbf{F}\beta = A_{[-1]} P_{\gamma_\beta}, \tag{2.26}$$

and extended to $\mathbf{F}$ *by requiring that* $\mathbf{F}$ *preserves the products, where* P_{γ_α}*,* P_{γ_β} *are as described in Def. 2.34.*

Remark 2.36. *The choice of an injective map* $F : \mathcal{F}tess_{mark} \to \mathcal{F}tess_{dot}$ *and the requirement* (2.19) *completely determines* $\mathbf{F}$*. In particular, we are forced to have* (2.26)*, and there's no need to check if* $\mathbf{F}\alpha$ *and* $\mathbf{F}\beta$ *satisfy the relations satisfied by* α *and* β*.*

It is quite easy to see that this map $\mathbf{F} : G_{mark} \to G_{dot}$ is injective, and that (the Kashaev group) G_{dot} is strictly larger than the image of (the Ptolemy-Thompson group) G_{mark}. In fact G_{dot} is much larger, so one might ask what characterizes the small subgroup $\mathbf{F}(G_{mark})$ (this question of characterization was raised by Frenkel [Fr] to the author). Actually, the map $\mathbf{F} : G_{mark} \to G_{dot}$ (2.17) can be naturally characterized as follows, instead of by requiring (2.19), which depends on the choice of F. Any $g \in G_{mark}$ is induced by the action on τ^*_{mark}

of an asymptotically rigid homeomorphism φ_g of $\mathbb{D}$ (see Prop. 2.19 and Def. 2.13). Applying φ_g to the standard dotted tessellation τ^*_{dot} yields a dotted tessellation which we denote here by $\varphi_g.\tau^*_{dot}$; one can observe that $\varphi_g.\tau^*_{dot}$ has the same configuration as τ^*_{dot} (see Def. 2.14). Now let $\mathbf{F}g$ be the unique element of G_{dot} corresponding to the morphism $[\tau^*_{dot}, \varphi_g.\tau^*_{dot}]$ of Pt_{dot} (see Prop. 2.28 and Def. 2.27). It is then easy to see:

Proposition 2.37. *The elements of G_{dot} induced by asymptotically rigid homeomorphisms of $\mathbb{D}$ (see Def. 2.13) applied to the standard dotted tessellation τ^*_{dot} (see Def. 2.9) form a subgroup of G_{dot}, and coincide with $\mathbf{F}(G_{mark})$.*

One can also obtain the following characterization of $\mathbf{F}(G_{mark})$ in G_{dot}:

Proposition 2.38. *The subgroup of G_{dot} of all elements which can be applied to τ^*_{dot} and preserve the configuration of it coincides with $\mathbf{F}(G_{mark})$ (see Def. 2.14).*

In the subsequent sections we study the projective representations of the groups G_{mark} and G_{dot} obtained from the quantum (universal) Teichmüller theory.

3. The dilogarithmic representations

In this section, we review the constructions of the quantization of the universal Teichmüller space done by Chekhov-Fock and by Kashaev. We will formulate their results in terms of the ‘dilogarithmic’ projective representations of the group $G_{mark} \cong T$ (which is the ‘asymptotically rigid mapping class group’ of $\mathbb{D}$, playing the role of a ‘universal’ mapping class group).

3.1. **Quantum universal Teichmüller space.** Quantization of the Teichmüller spaces for punctured (bordered) Riemann surfaces (of finite type) was first accomplished by Kashaev [Kas1] and independently by Chekhov-Fock [Fa] [CFo], in late 90’s. The relationship between the quantum Teichmüller theoy and the previous section is as follows. The coordinate system of the Teichmüller spaces used in the Chekhov-Fock quantization is the shear coordinates which go back to Thurston [Th]. In order to specify the shear coordinates, it is necessary to triangulate the surface into ideal triangles (then the shear coordinate for each ideal arc records how the two ideal triangles having that arc as a common side are glued to each other), that is, each object of Pt gives rise to a coordinate system, hence to a commutative topological $*$-algebra over $\mathbb{C}$, generated by the real-valued coordinate functions. Here a topological $*$-algebra over $\mathbb{C}$ means a topological vector space over the topological field $\mathbb{C}$ with a continuous product structure, and also with a $*$-structure compatible with the complex conjugation of $\mathbb{C}$. Each morphism, i.e. a change of triangulation, induces a coordinate change, and so we obtain the covariant functor

$$\beta : Pt \to \mathrm{Comm}, \tag{3.1}$$

(see Def. 2.23 for Pt) where Comm is the category of commutative topological $*$-algebras over $\mathbb{C}$.

The Teichmüller space also comes equipped with a canonical Poisson structure due to Weil and Petersson ([A]), and it is this Poisson manifold that one would wish to quantize. Furthermore, the coordinate changes induced by changes of triangulations leave this Poisson structure invariant, hence one would also wish that the quantum version of these coordinate changes still preserve the structure that replaces the Poisson structure in the quantum setting (namely, the product structure of the noncommutative algebras). For the universal Teichmüller space, denoted by $\mathcal{T}(1)$, all this can be formulated roughly as follows. We review here what’s written in [FuS]. First we need to recall the definition of a projective functor.

Definition 3.1. *To any category $\mathcal{C}$ whose morphisms are $\mathbb{C}$-vector spaces, one associates its* projectivization *$P\mathcal{C}$ having the same objects and new morphisms given by $Hom_{P\mathcal{C}}(C_1, C_2) = Hom_{\mathcal{C}}(C_1, C_2)/U(1)$, for any two objects C_1, C_2 of $\mathcal{C}$. Here $U(1) \subset \mathbb{C}$ acts by scalar multiplication. A* projective functor *into $\mathcal{C}$ is actually a functor into $P\mathcal{C}$.*

Definition 3.2. *Let A^* be the* category of topological $*$-algebras*. Two functors $F_1, F_2 : \mathcal{C} \to \mathrm{A}^*$* essentially coincide *if there exists a third functor F and natural transformations $F_1 \to F$, $F_2 \to F$ providing dense inclusions $F_1(O) \hookrightarrow F(O)$ and $F_2(O) \hookrightarrow F(O)$, for any object O of $\mathcal{C}$.*

Definition 3.3. *A* quantization *$\mathcal{T}^h$ of the universal Teichmüller space $\mathcal{T}(1)$ is a family of covariant functors $\beta^h : Pt \to \mathrm{A}^*$, depending smoothly on the real parameter h such that:*

1) *The limit $\lim_{h\to 0} \beta^h = \beta^0$ exists and essentially coincide with the functor β.*
2) *The product structure $\star$ of the non-commutative algebra $\beta^h(\tau)$ for each $\tau \in Pt$ satisfies*

$$f \star g = fg + h\{f, g\} + o(h),$$

where $\{,\}$ is the (Weil-Petersson) Poisson bracket on the space of functions on $\mathcal{T}(1)$.

As is usually the case for quantization of a physical system in quantum mechanics, the result of quantization of $\mathcal{T}(1)$ can be formulated as:

Definition 3.4. *A* projective $*$-representation *of the quantized universal Teichmüller space $\mathcal{T}^h$, specified by the functor $\beta^h : Pt \to A^*$, consists of the following data.*

(1) *A projective functor $Pt \to$* Hilb *to the category of Hilbert spaces. In particular, one associates a Hilbert space $\mathcal{L}_\tau$ to each tessellation τ and a unitary operator $\mathbf{K}_{[\tau,\tau']} : \mathcal{L}_\tau \to \mathcal{L}_{\tau'}$ to each morphism $[\tau, \tau']$ of Pt, defined up to a complex scalar of modulus 1.*
(2) *A $*$-representation ρ_τ of the Heisenberg algebra H^h_τ in the Hilbert space $\mathcal{L}_\tau$, such that the operators $\mathbf{K}_{[\tau,\tau']}$ intertwine the representations ρ_τ and $\rho_{\tau'}$, that is,*

$$\rho_\tau(w) = \mathbf{K}^{-1}_{[\tau,\tau']}\, \rho_{\tau'}(\beta^h([\tau, \tau'])(w))\, \mathbf{K}_{[\tau,\tau']}, \quad w \in H^h_\tau. \tag{3.2}$$

A projective $*$-representation of the quantized universal Teichmüller space $\mathcal{T}^h$ provides a projective representation of the Ptolemy groupoid Pt. By identiying the Hilbert spaces $\mathcal{L}_\tau$ for all τ, we can deduce a projective representation of the Ptolemy-Thompson group $T \cong G_{mark}$.

There's certainly a lot more to be said, so readers should consult the relevant references for more details (e.g. exposition in [FuS] and references therein). We will just state in §3.2 some consequences of the resulting projective representations of the Ptolemy-Thompson group obtained from Chekhov-Fock's quantization (or equivalent result of Fock-Goncharov's work [FoG] in terms of the quantum cluster varieties) which is used in [FuS]. However we will review Kashaev's quantization in §3.3 in more detail, as it is the version of the quantization of the universal Teichmüller space that we mainly use in the present paper. In fact, for Kashaev's quantization, we replace Pt in the above formulation by Pt_{dot}, as shall become clear in §3.3.

Both Chekhov-Fock's (or Fock-Goncharov's) and Kashaev's quantization make use of a special function named 'quantum dilogarithm' in a crucial manner (both β^h and $\mathbf{K}$ involve quantum dilogarithm); see (3.21) for the version used in the present paper. Therefore Funar and Sergiescu [FuS] called the projective representation of $T \cong G_{mark}$ resulting from the Chekhov-Fock quantization the '*dilogarithmic (projective) representation*'. However, in the present paper, the word 'dilogarithmic representation' or 'dilogarithmic central extension' apply both for the Chekhov-Fock construction and the Kashaev construction, because Kashaev also used the quantum dilogarithm function in his result (i.e. in his projective representation).

3.2. **The dilogarithmic representation of G_{mark} via the Chekhov-Fock (or Fock-Goncharov) quantization.** Instead of trying to state the construction or the result of the Chekhov-Fock quantization (or Fock-Goncharov's work), we only state one of their consequences. The quantization yields a projective representation $G_{mark} \to PGL(V)$, or more precisely an 'almost linear representation' (in order to distinguish with *projective* representation), i.e. a map

$$\rho^{CF} : G_{mark} \to GL(V), \quad \text{for} \quad V = L^2(\mathbb{R}^{\tau^{(1)}}), \tag{3.3}$$

which is not necessarily a group homomorphism but becomes one after being composed with the canonical map $GL(V) \to PGL(V)$ (see below, and §4.1 for 'almost linear representation'), where $GL(V)$ above can be replaced by $U(V)$, the group of unitary operators on V. Here, $\tau^{(1)}$, the set of ideal arcs of a (Farey-type) tessellation, can be naturally identified by $\mathbb{Q} \setminus \{0, 1\}$ as mentioned in Rem. 2.10. In fact, to be even more precise, the data we obtain from the quantization is an 'almost G_{mark}-homomorphism', i.e. a group homomorphism

$$\rho^{CF} : F_{mark} \to GL(V), \quad \text{where} \quad V = L^2(\mathbb{R}^{\tau^{(1)}}), \tag{3.4}$$

rather than (3.3), where

$$G_{mark} = F_{mark}/R_{mark}, \tag{3.5}$$

where F_{mark} is the free group generated by α, β and R_{mark} its normal subgroup generated by the relations of α, β for G_{mark} (in (2.9)). The group homomorphism (3.4) being an almost G_{mark}-homomorphism means $\rho^{CF}(R_{mark}) \subset \mathbb{C}^*$ (see Def. 4.1), so that the composition of ρ^{CF} and the canonical projection $GL(V) \to PGL(V)$ induces a group homomorphism $G_{mark} \to PGL(V)$, which is usually called a *projective representation* of G_{mark}.

As is well known and reviewed and generalized in §4.1, the data (3.4) (almost linear representation of G_{mark}) induces a central extension of G_{mark}, denoted by $\widehat{G}^{CF}_{mark}$, resolving the almost linear representation of G_{mark} to a genuine representation $\widehat{G}^{CF}_{mark} \to GL(V)$ of the group $\widehat{G}^{CF}_{mark}$. In [FuS], they use the name T for the group G_{mark}, denote by $\widehat{T}$ the central extension $\widehat{G}^{CF}_{mark}$, and call it the 'dilogarithmic extension' (the name is because the almost linear representation ρ^{CF} involves the quantum dilogarithm function). One of the main results of [FuS] is the following (see also Thm. 4.8 of the present paper):

Proposition 3.5 ([FuS])**.** *The dilogarithmic extension $\widehat{T}$ is identified with T^*_{ab}.*

They computed the presentation of $\widehat{T}$ in terms of generators and relations, and identified it with the central extension $T_{1,0,0,0}$ of T which appears in Thm. 4.7 of the present paper. And they noticed that there is a central extension of T denoted by T^*_{ab} which has a 'geometric meaning' which has a same presentation as $T_{1,0,0,0}$, hence concluding the above Prop. 3.5. The construction of the group T^*_{ab} and its identification with $\widehat{T} \cong \widehat{G}^{CF}_{mark}$ will be reviewed and discussed in §5.1 and §5.2.

3.3. **The dilogarithmic representation of G_{mark} via the Kashaev quantization.** Let us briefly review Kashaev's quantization of the universal Teichmüller space. Denote by $\mathcal{T}(1)$ the universal Teichmüller space. Readers can also consult Kashaev's original paper [Kas1], or for an exposition, see e.g. Teschner [Te] or Guo-Liu [GuLi]. Suppose we have a fixed choice of a horocycle at each puncture (vertices of τ), as it's necessary to do so.

Definition 3.6 (Kashaev's coordinates of the universal Teichmüller space $\mathcal{T}(1)$)**.** *In Kashaev's quantization of the universal Teichmüller space $\mathcal{T}(1)$, each choice of a dotted tessellation $\tau_{dot} \in \mathcal{F}tess_{dot}$ (see Def. 2.9) gives rise to a coordinate system of $\mathcal{T}(1)$ which assigns to each triangle*

$j \in \tau^{(2)} = \mathbb{Q}^\times$ the two coordinates p_j, q_j, i.e. an injective map $\mathcal{T}(1) \to (\mathbb{R}^{\tau^{(2)}})^2 = (\mathbb{R}^{\mathbb{Q}^\times})^2$, where

$$p_j = \ell_{j,1} - \ell_{j,2}, \quad q_j = \ell_{j,3} - \ell_{j,2}, \tag{3.6}$$

called the Kashaev coordinates*, where $\ell_{j,1}, \ell_{j,2}, \ell_{j,3}$ are the geodesic lengths of the sides of the triangle j where the cyclic labeling of the three sides of each triangle is determined by the choice of the distinguished corner (or dot $\bullet$), where we trim the sides using the chosen horocycles (so some length ℓ can be negative); see Fig. 11. These ℓ's are the logarithm of the 'lambda lengths' of Penner (see e.g.* [P1] *or* [P2]*).*

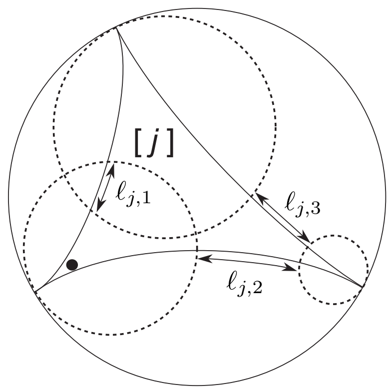


FIGURE 11. The lambda lengths for the triangle j; here $\ell_{j,1} < 0$, $\ell_{j,2} > 0$, $\ell_{j,3} > 0$, and the dotted circles are the chosen horocycles at the vertices

Denote by $\{\cdot,\cdot\}$ the canonical Weil-Petersson Poisson bracket on the algebra of functions on $\mathcal{T}(1)$.

Proposition 3.7. *The Kashaev's coordinate functions satisfy*

$$\{p_j, q_k\} = \delta_{j,k}, \quad \{p_j, p_k\} = \{q_j, q_k\} = 0, \quad \forall j, k \in \tau^{(2)} = \mathbb{Q}^\times, \tag{3.7}$$

where $\delta_{j,k} = 1$ when $j = k$ and 0 when $j \neq k$. This system has a canonical quantization

$$p_j \to \hat{p}_j = 2\pi b P_j, \quad q_j \to \hat{q}_j = 2\pi b Q_j, \tag{3.8}$$

realized as self-adjoint operators on (a dense subspace of) the Hilbert space

$$\mathscr{M} = L^2(\mathbb{R}^{\tau^{(1)}}) = L^2(\mathbb{R}^{\mathbb{Q}^\times}) \tag{3.9}$$

(where any element of $\mathscr{M}$ is a function in the variable $\mathbf{x} = (x_j)_{j \in \mathbb{Q}^\times}$), where $b \in \mathbb{R}$ is the generic quantization parameter (so that $b^2 \notin \mathbb{Q}$), and the operators P_j, Q_j on $\mathscr{M}$ are given by

$$P_j f = \frac{1}{2\pi i} \frac{\partial}{\partial x_j} f, \quad Q_j f = x_j f, \quad \textit{for } f \in \mathscr{M} = L^2(\mathbb{R}^{\mathbb{Q}^\times}), \tag{3.10}$$

satisfying $[P_j, Q_k] = \frac{1}{2\pi i}\delta_{j,k}$, $[P_j, P_k] = [Q_j, Q_k] = 0$ (the Heisenberg algebra). Then one has

$$[\hat{p}_j, \hat{q}_k] = -2\pi i b^2 \delta_{j,k}, \quad [\hat{p}_j, \hat{p}_k] = [\hat{q}_j, \hat{q}_k] = 0. \tag{3.11}$$

Usually, quantization of the space $\mathcal{T}(1)$ is described as a family of non-commutative algebras depending on a real parameter, whose generators are realized as self-adjoint operators on a Hilbert space. For our case, we use the exponents

$$\hat{Y}_j = e^{\hat{q}_j}, \quad \hat{Z}_j = e^{\hat{p}_j} \tag{3.12}$$

of $\hat{p}_j, \hat{q}_j$ as the generators of the non-commutative algebra, subject to the relations (3.11).

Definition 3.8. *For $b \in \mathbb{R}$, $b^2 \notin \mathbb{Q}$, define $q \in \mathbb{C}^*$ by*

$$q = e^{\pi i b^2}. \tag{3.13}$$

For $\tau_{dot} \in \mathcal{F}tess_{dot}$ (which in particular gives a bijection between triangles $\tau^{(2)}$ and $\mathbb{Q}^\times$), let the Kashaev algebra $\mathcal{K}^q_{\tau_{dot}}$ *be the algebra generated by $\hat{Y}_j, \hat{Z}_j$, $j \in \mathbb{Q}^\times$ with the relations*

$$\hat{Y}_j \hat{Z}_j = q^2 \hat{Z}_j \hat{Y}_j, \quad [\hat{Y}_j, \hat{Z}_k] = [\hat{Y}_j, \hat{Y}_k] = [\hat{Z}_j, \hat{Z}_k] = 0 \quad \textit{for } j \neq k. \tag{3.14}$$

Elements of $\mathcal{K}^q_{\tau_{dot}}$ can be thought of as operators on a Hilbert space $\mathscr{M} = L^2(\mathbb{R}^{\mathbb{Q}^\times})$ (3.9) *via the representation π*

$$\pi(\hat{Y}_j) = e^{2\pi b Q_j}, \quad \pi(\hat{Z}_j) = e^{2\pi b P_j}, \tag{3.15}$$

where P_j, Q_j are as defined in (3.10).

The Kashaev algebra $\mathcal{K}^q_{\tau_{dot}} = \langle \hat{Y}_j, \hat{Z}_j : j \in \mathbb{Q}^\times \rangle / (\text{rels in (3.14)})$ is the non-commutative deformation under this quantization of the algebra of functions on $\mathcal{T}(1)$ generated by the (exponents of the) coordinate functions $Y_j = e^{p_j}$, $Z_j = e^{q_j}$ (which depend on τ_{dot}).

For a finite type surface, the Weil-Petersson Poisson structure on the Teichmüller space of the surface is preserved under the action of the mapping class group of the surface (group isotopy classes of orientation-preserving homeomorphisms of the surface permuting the punctures and the distinguished points on the boundary respectively). For the case of the universal Teichmüller space $\mathcal{T}(1)$, each element of 'the universal mapping class group' is an asymptotically rigid homeomorphism of $\mathbb{D}$ (see Def. 2.13 and Prop. 2.19), hence can be represented as an element of G_{dot} (changes of dotted tessellations) from its action on τ^*_{dot} (see §2.2 of the present paper). Each element of G_{dot}, i.e. a change of dotted tessellations, yields a corresponding change of Kashaev's coordinates of $\mathcal{T}(1)$. It turns out that the coordinate change maps of $\mathcal{T}(1)$ induced by the action of G_{dot} preserve the Weil-Petersson Poisson structure of $\mathcal{T}(1)$. Therefore we would want to construct a quantization such that G_{dot} still 'acts' on the non-commutative algebra $\mathcal{K}^q_{\tau_{dot}}$, preserving the algebra structure. If we can identify $\mathcal{K}^q_{\tau_{dot}}$ for different dotted tessellations, this would mean that we want G_{dot} to act on $\mathcal{K}^q_{\tau_{dot}}$ as algebra automorphisms.

For each element of G_{dot} which represents the morphism $[\tau_{dot}, \tau'_{dot}]$ of Pt_{dot} (see Def. 2.27), Kashaev constructed a map $\mathcal{K}^q_{\tau_{dot}} \to \mathcal{K}^q_{\tau'_{dot}}$ which yields the corresponding coordinate change in the classical limit, and which after a canonical identification of $\mathcal{K}^q_{\tau_{dot}}$ and $\mathcal{K}^q_{\tau'_{dot}}$ becomes an algebra automorphism of $\mathcal{K}^q_{\tau_{dot}}$. To be more precise, his result is in terms of the representation π (3.15) rather than the algebra $\mathcal{K}^q_{\tau_{dot}}$ itself. We first describe the canonical identification of $\mathcal{K}^q_{\tau_{dot}}$ and $\mathcal{K}^q_{\tau'_{dot}}$.

Definition 3.9. *Suppose $\tau_{dot}, \tau'_{dot} \in \mathcal{F}tess_{dot}$, whose triangle-labeling rules let us identify $\tau^{(2)}$ with $\{j : j \in \mathbb{Q}^\times\}$ and $(\tau')^{(2)}$ with $\{j' : j \in \mathbb{Q}^\times\}$. Define the map $I_{\tau_{dot}, \tau'_{dot}} : \mathcal{K}^q_{\tau_{dot}} \to \mathcal{K}^q_{\tau'_{dot}}$ by*

$$I_{\tau_{dot}, \tau'_{dot}} : \hat{Y}_j \longmapsto \hat{Y}_{j'}, \quad \hat{Z}_j \longmapsto \hat{Z}_{j'}, \quad \forall j \in \mathbb{Q}^\times, \tag{3.16}$$

which is easily seen to be an algebra isomorphism in view of (3.14).

Now one of the main result of Kashaev on the quantum (universal) Teichmüller space can be written as follows.

Theorem 3.10. *Let the Kashaev group $G_{\mathbb{Q}^\times} = G_{dot}$ (see Def. 2.30) be given as a finitely presented group $G_{dot} = F_{dot}/R_{dot}$ for F_{dot} the free group generated by $A_{[j]}$, $T_{[j][k]}$, $P_{(jk)}$ ($j,k \in \mathbb{Q}^\times$, $j \neq k$) and R_{dot} its normal subgroup generated by the relations for G_{dot}, as in (2.16). For the Hilbert space $\mathscr{M} = L^2(\mathbb{R}^{\mathbb{Q}^\times})$ (3.9), consider the group homomorphism*

$$\rho : F_{dot} \to GL(\mathscr{M}) \tag{3.17}$$

given by assigning to each generator F_{dot} a unitary operator on $\mathscr{M}$ as follows:

$$\rho(A_{[j]}) = e^{-\pi i/3} e^{3\pi i Q_j^2} e^{\pi i (P_j + Q_j)^2}, \tag{3.18}$$

$$\rho(T_{[j][k]}) = e^{2\pi i P_j Q_k} \Psi_b(Q_j + P_k - Q_k)^{-1}, \tag{3.19}$$

$$(\rho(P_{(jk)})f)(\ldots, x_j, \ldots, x_k, \ldots) = f(\ldots, x_k, \ldots, x_j, \ldots), \tag{3.20}$$

where P_j, Q_j are as in (3.10), and Ψ_b is one version of the so-called (non-compact) quantum dilogarithm function which is defined by the integral formula

$$\Psi_b(z) = \exp\left(\frac{1}{4} \int_{\Omega_0} \frac{e^{-2izw} dw}{\sinh(wb)\sinh(w/b)w} \right) \tag{3.21}$$

first in the strip $|\mathrm{Im}\, z| < (b + b^{-1})/2$, where Ω_0 means the real line contour with a detour around 0 (origin) along a small half circle above the real line, and analytically continued to a meromorphic function on the complex plane using the following functional equations:

$$\left\{ \begin{array}{l} \Psi_b(z - ib/2) = (1 + e^{2\pi b z}) \Psi_b(z + ib/2), \\ \Psi_b(z - ib^{-1}/2) = (1 + e^{2\pi b^{-1} z}) \Psi_b(z + ib^{-1}/2). \end{array} \right. \tag{3.22}$$

Then the quantum version of the coordinate change induced by an element g of G_{dot} (more precisely $g \in F_{dot}$) is given by the conjugation by $\rho(g)$, in the following sense. Suppose $\tau_{dot} \in \mathcal{F}tess_{dot}$. For each $g \in G_{dot}$ which can be applied to τ_{dot}, we associate an algebra isomorphism

$$\Phi_g^q : \mathcal{K}^q_{\tau_{dot}} \to \mathcal{K}^q_{g.\tau_{dot}} \tag{3.23}$$

which after identification of $\mathcal{K}^q_{g.\tau_{dot}}$ with $\mathcal{K}^q_{\tau_{dot}}$ via (3.16) becomes as follows, in terms of the representation π of $\mathcal{K}^q_{g.\tau_{dot}}$:

$$\pi\left(I_{g.\tau_{dot}, \tau_{dot}} \circ \Phi_g^q \right) : \begin{array}{ccl} \pi(\mathcal{K}^q_{\tau_{dot}}) & \longrightarrow & \pi(\mathcal{K}^q_{\tau_{dot}}) \\ \pi(\hat{u}) & \longmapsto & \rho(g)\pi(\hat{u})\rho(g)^{-1}, \quad \forall \hat{u} \in \mathcal{K}^q_{\tau_{dot}}. \end{array} \tag{3.24}$$

The map (3.24) is well-defined, and obviously provides an algebra isomorphism of $\pi(\mathcal{K}^q_{\tau_{dot}})$ because it's a conjugation. When $q = e^{\pi i b^2} \to 1$ (via $b \to 0$ in $\mathbb{R}$), the limit of the map (3.24) recovers the classical coordinate change map induced by g.

Let us explain how the operators (3.18) and (3.19) act on the L^2 space. Since $\rho(A_{[j]})$ involves only P_j and Q_j (and no P_k, Q_k for $k \neq j$), we can think of the RHS of (3.18) as an operator just on $L^2(\mathbb{R}, dx_j)$, and then this operator can be written as

$$L^2(\mathbb{R}, dx_j) \ni f(x_j) \longmapsto (\mathbf{A}f)(x_j) = e^{-\pi i/12} \int_{\mathbb{R}} e^{2\pi i y_j x_j} e^{\pi i x_j^2} f(y_j) dy_j \in L^2(\mathbb{R}, dx_j), \tag{3.25}$$

which makes sense first for nice functions f, say in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, or in the Schwartz space, then can be extended to the whole $L^2(\mathbb{R})$ by continuity. If we denote this operator on $L^2(\mathbb{R}, dx)$

by $\mathbf{A}$ for convenience, then it is the unique unitary operator up to the scalar multiplication by a complex number of modulus one which satisfies

$$\mathbf{A}Q\mathbf{A}^{-1} = P - Q, \quad \mathbf{A}P\mathbf{A}^{-1} = -Q, \tag{3.26}$$

where $P = \frac{1}{2\pi i}\frac{d}{dx}$ and $Q = x$ are symmetric operators on $L^2(\mathbb{R}, dx)$. The above equations (3.26) (from which it is almost immediate that $\mathbf{A}^3 = id$) can still be written as, for example,

$$\mathbf{A}e^{isQ}\mathbf{A}^{-1} = e^{is(P-Q)}, \quad \mathbf{A}e^{isP}\mathbf{A}^{-1} = e^{-isQ}, \quad s \in \mathbb{R}, \tag{3.27}$$

which again translate to

$$\mathbf{A} : e^{isx}f(x) \longmapsto e^{s^2/(4\pi i)}e^{-isx}(\mathbf{A}f)(x + \frac{s}{2\pi}), \qquad f(x + \frac{s}{2\pi}) \longmapsto e^{-isx}(\mathbf{A}f)(x). \tag{3.28}$$

This kind of operators that can be written as the exponential of quadratic expressions in P and Q are analogues of the Fourier transformation $\mathcal{F} : f(x) \mapsto \int_{\mathbb{R}} e^{-2\pi ixy}f(y)dy$, which is characterized up to a multiplicative constant by $\mathcal{F}Q\mathcal{F}^{-1} = -P$ and $\mathcal{F}P\mathcal{F}^{-1} = Q$.

For (3.19), we view its RHS as an operator acting on $L^2(\mathbb{R}^2, dx_jdx_k)$. The unitary operator $e^{2\pi iP_jQ_k}$ acts as

$$e^{2\pi iP_jQ_k} : f(x_j, x_k) \longmapsto f(x_j + x_k, x_k). \tag{3.29}$$

One way to explain the second part of the RHS of (3.19) is as follows. First, write $\Psi_b(Q_j + P_k - Q_k)^{-1} = \mathbf{A}_k\Psi_b(Q_j + Q_k)^{-1}\mathbf{A}_k^{-1}$, using (3.26). We know how unitary operators $\mathbf{A}_k, \mathbf{A}_k^{-1}$ act. The operator $\Psi_b(Q_j + Q_k)^{-1}$ is just multiplication by $\Psi_b(x_j + x_k)^{-1}$ (for $x \in \mathbb{R}$, the complex number $\Psi_b(x)$ is of modulus 1).

By Schur's lemma due to the irreducibility of the representation π, we should expect from Thm 3.10 that ρ generates a projective representation of G_{dot} on $\mathscr{M}$. More concretely, we have the following result:

Proposition 3.11 ([Kas3])**.** *The map ρ (3.17) defined in (3.18), (3.19) and (3.20) satisfies*

$$\rho(A_{[j]})^3 = id, \tag{3.30}$$

$$\rho(T_{[k][\ell]})\,\rho(T_{[j][k]}) = \rho(T_{[j][k]})\,\rho(T_{[j][\ell]})\,\rho(T_{[k][\ell]}), \tag{3.31}$$

$$\rho(A_{[j]})\,\rho(T_{[j][k]})\,\rho(A_{[k]}) = \rho(A_{[k]})\,\rho(T_{[k][j]})\,\rho(A_{[j]}), \tag{3.32}$$

$$\rho(T_{[j][k]})\,\rho(A_{[j]})\,\rho(T_{[k][j]}) = \zeta\,\rho(A_{[j]})\,\rho(A_{[k]})\,\rho(P_{(jk)}), \tag{3.33}$$

where $j, k, \ell \in \mathbb{Q}^\times$ are mutually distinct, and

$$\zeta = e^{-\pi i(b+b^{-1})^2/12}, \tag{3.34}$$

as well as the trivial relations:

$$\rho(P_{(jk)})^2 = id, \quad \rho(P_{(jk)})\,f_{\ldots,[j],\ldots,[k],\ldots}\,\rho(P_{(jk)}) = f_{\ldots,[k],\ldots,[j],\ldots}, \quad \rho(P_{(jk)}) = \rho(P_{(kj)}), \tag{3.35}$$

where $f_{\ldots,[j],\ldots,[k],\ldots}$ is any word in $\{\rho(g) : g \in \{\text{generators of } G_{dot}\}\}$ (conjugation by $\rho(P_{(jk)})$ results in exchanging the subscripts $[j]$ and $[k]$), and any two words in $\{\rho(g) : g \in \{\text{generators of } G_{dot}\}\}$ whose collections of subscripts don't intersect with each other commute.

Hence $\rho : F_{dot} \to GL(\mathscr{M})$ (3.17) defined in Thm 3.10 is in fact an 'almost G_{dot}-homomorphism' into $GL(\mathscr{M})$ in the sense of Def. 4.1, i.e. we have $\rho(R_{dot}) \subset \mathbb{C}^*$.

Hence, as in §4.1, the map ρ (3.17), when considered as a map from G_{dot} to $GL(\mathscr{M})$, can also be called an 'almost linear representation' of G_{dot} on $\mathscr{M}$, as done also in [FuS]. In particular, ρ induces a projective representation of the group G_{dot} on $\mathscr{M}$, i.e. a group homomorphism $G_{dot} \to PGL(\mathscr{M})$. We can pre-compose this map $\rho : F_{dot} \to GL(\mathscr{M})$ (3.17) of Kashaev with the map

$F_{mark} \to F_{dot}$ that is given by the same formula as $\mathbf{F} : G_{mark} \to G_{dot}$ (2.17) (hence inducing $\mathbf{F}$ by modding out R_{mark} and R_{dot} respectively) to obtain the almost G_{mark}-homomorphism $\rho^{Kash} : F_{mark} \to GL(\mathcal{M})$ of G_{mark} (so, roughly speaking, we will let '$\rho^{Kash} := \rho \circ \mathbf{F}$'):

Corollary 3.12. *Write $G_{mark} = F_{mark}/R_{mark}$ by generators and relations as in* (3.5)*. Define the group homomorphism*

$$\rho^{Kash} : F_{mark} \to GL(\mathscr{M}) \tag{3.36}$$

by assigning the following images to the free generators (see Thm. 3.10 for the definition of ρ)

$$\rho^{Kash}(\alpha) := \rho(A_{[-1]})\rho(T_{[-1][1]})^{-1}\rho(A_{[1]})\rho(P_{\gamma_\alpha}), \tag{3.37}$$

$$\rho^{Kash}(\beta) := \rho(A_{[-1]})\rho(P_{\gamma_\beta}). \tag{3.38}$$

Then this map ρ^{Kash} (3.36) *is an almost G_{mark}-homomorphism in the sense of Def. 4.1, which induces an 'almost linear representation' $G_{mark} \to GL(\mathscr{M})$ (Def. 4.1), hence a projective representation $G_{mark} \to PGL(\mathscr{M})$.*

In the following section, we compare the two almost linear representations (or almost G_{mark}-homomorphisms, to be more precise) ρ^{CF} and ρ^{Kash} of the group G_{mark}.

4. The relationship between the two representations, and the induced central extensions

In this section, we take the projective representations (more precisely, the 'almost linear representations', or the 'almost G_{mark}-homomorphisms') ρ^{CF} and ρ^{Kash} of G_{mark}, compute the two central extensions of G_{mark} which they yield, and study the difference between them.

4.1. Classification of the central extensions of G_{mark}.

We first study some constructions of the central extensions of groups. It is well known (e.g. as pointed out in [FuS]) that a projective representation

$$\eta : G \to PGL(V) \tag{4.1}$$

(where η is a genuine group homomorphism) of a group G on a vector space V gives rise to a central extension $\widetilde{G}$ of G, as a pullback of the canonical projection $p : GL(V) \to PGL(V)$.

However, the data for a projective representation is often given as an 'almost linear representation' (to distinguish with projective representation as in (4.1))

$$\bar{\eta} : G \to GL(V), \tag{4.2}$$

which a priori isn't a group homomorphism, while the composition $\pi \circ \bar{\eta}$ with the projection map $\pi : GL(V) \to PGL(V)$ is. There is a procedure (see e.g. [FuS]) to obtain a central extension $\widehat{G}$ of G out of this data, the smallest one such that the induced map $\widehat{G} \to GL(V)$ is a genuine group homomorphism. Then $\widehat{G}$ is a subgroup of $\widetilde{G}$ (called a 'minimal reduction' of $\widetilde{G}$ in Funar-Kashaev [FuKas]), and we usually are interested in these (minimally reduced) extensions $\widehat{G}$ in the classification of the central extensions of G.

For our later use in the present paper, we formulate here a recipe for cooking up a central extension from a little bit more general setting (generalizing the almost linear representations on vector spaces).

Definition 4.1. *Let G be a group presented by generators and relations, i.e. $G = F/R$ where F is a free group (for generators) and R is the normal subgroup of F generated by the relations. Let H be a group. Now, a group homomorphism*

$$\bar{\eta} : F \to H \tag{4.3}$$

is said to be an almost G-homomorphism *if $\bar{\eta}(R)$ is contained in the center of $\bar{\eta}(F)$.*

When $H = GL(V)$ for some vector space V, we call such $\bar{\eta}$ (4.3) an 'almost linear representation of G on V'. In this case, by abuse of notation, we also call the map (4.2) an 'almost linear representation' (but the map (4.2) is not necessarily a group homomorphism).

A short way of stating the construction of the central extension of the group G is as follows. Given an almost G-homomorphism $\bar{\eta} : F \to H$ (where $G = F/R$ as in Def. 4.1), we get a central extension $\widehat{G} = F/(R \cap \ker \bar{\eta})$ of G by (the group isomorphic to) $\bar{\eta}(R)$. However, this construction isn't really useful in presenting the resulting central extension $\widehat{G}$ by generators and relations.

We can construct $\widehat{G}$ more concretely, as follows. Let Z be a group isomorphic to $\bar{\eta}(R)$ and let us fix an isomorphism $\phi : \bar{\eta}(R) \to Z$. Now we consider the free product $F * Z$, and let R' be its normal subgroup generated by $r(\phi(\bar{\eta}(r)))^{-1}$ (called the *lifted relations*) for r the generators of R and $[f, z]$ for f, z the generators of F, Z, respectively. This yields a central extension

$$\widehat{G} = F * Z/R' \tag{4.4}$$

of G. Using a presentation of Z by generators and relations, we easily obtain the presentation for $\widehat{G}$ by generators and relations (lifted relations, commuting relations, and relations for the center Z). Moreover, we also obtain the natural lift of the original generators, by the group homomorphism

$$\Psi : F \to F * Z/R' \tag{4.5}$$

induced by the inclusion $F \to F * Z$.

For completeness, let us prove that $\widehat{G} = F/(R \cap \ker \bar{\eta})$ and $F * Z/R'$ are indeed isomorphic. To avoid confusion, denote $\widehat{G}_0 = F * Z/R'$ for the moment. As in (4.5) we have a group homomorphim $\Psi : F \to F * Z/R' = \widehat{G}_0$. It suffices to prove that Ψ is surjective, and that $\ker \Psi = R \cap \ker \bar{\eta}$. For surjectivity, we should just show $z \in \Psi(F)$ for any $z \in Z$. We know from the relations R' that for any $r \in R \subset F$ we have $r \equiv \phi(\bar{\eta}(r))$ in $\widehat{G}_0$. Since $\phi : \eta(R) \to Z$ is an isomorphism, for any $z \in Z$ there exists $r \in R$ such that $\phi(\bar{\eta}(r)) \equiv z$ in $\widehat{G}_0$. Then $\Psi(r) \equiv r \equiv \phi(\bar{\eta}(r)) \equiv z \in \widehat{G}_0$, so Ψ is surjective. Compose Ψ with the map $\widehat{G}_0 \to F/R$ which is quotienting by Z, to get $F \to F/R$ (which is easy to see), which coincides with just the natural projecting map from F to F/R. If $x \in \ker \Psi \subset F$, then x is mapped by this composed map into the identity element of F/R, meaning that $x \in R$. Then, again using the relations R' (which say $r \equiv \phi(\bar{\eta}(r))$ for any $r \in R$), we have $\Psi(x) \equiv x \equiv \phi(\bar{\eta}(x))$ in $\widehat{G}_0$. Now, $\Psi(x) \equiv 1$ if and only if $\bar{\eta}(x) = 1$, because $\phi : \bar{\eta}(R) \to Z$ is an isomorphism and the natural map $Z \to \widehat{G}_0$ is an injection ($\because$ it's not hard to see that the natural map from Z to the subgroup $R * Z/(R' \cap (R * Z))$ of $\widehat{G}_0$ is an isomorphism). Therefore $x \in \ker \bar{\eta}$, hence $\ker \Psi = \ker \bar{\eta} \cap R$, as desired.

We can formulate this construction of $\widehat{G} = F * Z/R'$ as follows. Let $G_1 := \bar{\eta}(F) \leq H$ and let $\pi : G_1 \to G_1/\text{center}(G_1)$ be the projection. We observe that $\pi \circ \bar{\eta} : F \to G_1/\text{center}(G_1) = \pi(G_1)$ induces a well-defined group homomorphism $G \to \pi(G_1)$, because it sends R to 1. Now define

$\widehat{G}$ as the pullback of G under the map $G_1 \to \pi(G_1)$:

$$\begin{tikzcd} \widehat{G} \arrow[r, dotted] \arrow[d, dotted] & G \arrow[d] \\ G_1 \arrow[r] & \pi(G_1), \end{tikzcd} \tag{4.6}$$

or the fiber product of $G \to \pi(G_1)$ and $G_1 \to \pi(G_1)$ (if we use $H \to H/\text{center}(H)$ in the bottom row of (4.6), we get a possibly bigger central extension $\widetilde{G}$, as an analog of pulling back $G \to PGL(V)$ along $GL(V) \to PGL(V)$).

If the homomorphism $G \to \pi(G_1)$ is actually an isomorphism (we may call such $\bar{\eta}$ a 'faithful' almost G-homomorphism), it's easy to show that $\widehat{G} \to G_1$ is also an isomorphism. Thus for *any* central extension $\widetilde{G}$ of G, by setting $H = \widetilde{G}$ and choosing a section $G \to \widetilde{G}$, we have the notion of a 'tautological' almost G-homomorphism $F \to \widetilde{G}$, and it's not difficult to see that the central extension of G obtained by the above procedure is isomorphic to $\widetilde{G}$:

Definition 4.2. *Let $G = F/R$ be as in Def. 4.1, and let $\widetilde{G}$ be a central extension of G. An almost G-homomorphism $\bar{\eta} : F \to \widetilde{G}$ is said to be* tautological *if $\bar{\eta}(F) = \widetilde{G}$ holds and the induced map $G \to \pi(G_1) \cong G$ as described above is the identity map.*

Proposition 4.3. *The central extension of G obtained by the above described procedure from a tautological almost G-homomorphism $F \to \widetilde{G}$ is isomorphic to $\widetilde{G}$.*

We can also introduce the notion of the equivalence of almost G-homomorphisms:

Definition 4.4. *Let $G = F/R$ be as in Def. 4.1 and H_1, H_2 be groups. Two almost G-homomorphisms $\bar{\eta}_1 : F \to H_1$, $\bar{\eta}_2 : F \to H_2$ are said to be* equivalent *if the subgroups $G_j := \bar{\eta}_j(F)$ of H_j (for $j = 1, 2$) are isomorphic to each other, and there is an isomorphism $\Phi_{12} : G_1 \to G_2$ such that $\Phi_{12} \circ \bar{\eta}_1 = \bar{\eta}_2$. We write this as*

$$(\bar{\eta}_1 : F \to H_1) \simeq_{\Phi_{12}} (\bar{\eta}_2 : F \to H_2). \tag{4.7}$$

It is easy to see the following two observations, which will be used later.

Proposition 4.5. *The equivalence of the almost G-homomorphisms is an equivalence relation. More importantly, any two equivalent almost G-homomorphisms yield isomorphic central extensions of G via the above procedure. Also, the map Φ_{12} in Def. 4.4 provides an explicit isomorphism between these two central extensions.*

Lemma 4.6. *Let $G, F, R, H_1, H_2, \bar{\eta}_1, \bar{\eta}_2, G_1, G_2, \Phi_{12}$ be as in Def. 4.4. Let $G' = F'/R'$ be another group presented as generators and relations. Then, if $\phi : F' \to F$ is a group homomorphism with $\phi(R') \subset R$, then the pre-compositions of $\bar{\eta}_1, \bar{\eta}_2$ with ϕ are equivalent almost G'-homomorphisms, i.e. $\bar{\eta}_1 \circ \phi \simeq \bar{\eta}_2 \circ \phi$, via (the appropriate restriction of) Φ_{12}.*

For a group H_1', suppose that $\psi_1 : G_1 \to H_1'$ is an injective group homomorphism. Then the post-composition of $\bar{\eta}_1$ with ψ_1, i.e. $\psi_1 \circ \bar{\eta}_1 : F \to H_1'$, is an almost G-homomorphism, and is equivalent to $\bar{\eta}_1 : F \to H_1$, via $\psi_1^{-1} : \psi_1(G_1) \to G_1$.

Coming back to our situation, both ρ^{CF} and ρ^{Kash} (see (3.4) and (3.36)) are almost G_{mark}-homomorphisms. As just seen, these yield central extensions of G_{mark}, presented as generators and relations. The main task of constructing these presentations is computing the lifted relations, using each of the two almost G_{mark}-homomorphisms.

Meanwhile, Funar and Sergiescu [FuS] classified the presentations of all possible central extensions of $T \cong G_{mark}$ by $\mathbb{Z}$, and computed the corresponding extension classes in $H^2(T) \cong H^2(G_{mark})$.

Theorem 4.7 (Funar-Sergiescu, Thm. 1.2 of [FuS]). *Let $T_{n,p,q,r}$ be the group presented by the generators $\bar{\alpha}$, $\bar{\beta}$, z and the relations*

$$(\bar{\beta}\bar{\alpha})^5 = z^n, \quad \bar{\alpha}^4 = z^p, \quad \bar{\beta}^3 = z^q, \tag{4.8}$$

$$[\bar{\beta}\bar{\alpha}\bar{\beta}, \, \bar{\alpha}^2\bar{\beta}\bar{\alpha}\bar{\beta}\bar{\alpha}^2] = z^r, \quad [\bar{\beta}\bar{\alpha}\bar{\beta}, \, \bar{\alpha}^2\bar{\beta}\bar{\alpha}^2\bar{\beta}\bar{\alpha}\bar{\beta}\bar{\alpha}^2\bar{\beta}^2\bar{\alpha}^2] = 1, \quad [\bar{\alpha}, z] = [\bar{\beta}, z] = 1. \tag{4.9}$$

Then each central extension of the Ptolemy-Thompson group $T \cong G_{mark}$ (see Thm 2.21) by $\mathbb{Z}$ is of the form $T_{n,p,q,r}$, for some $n, p, q, r \in \mathbb{Z}$. Moreover, the class $c_{T_{n,p,q,r}} \in H^2(T) \cong H^2(G_{mark})$ of the extension $T_{n,p,q,r}$ is given by

$$c_{T_{n,p,q,r}} = (12n - 15p - 20q - 60r)\chi + r\alpha, \tag{4.10}$$

where $\alpha, \chi \in H^2(T)$ are the so-called discrete Godbillon-Vey class and the Euler class, respectively (see [FuS] *and references therein, for more details about these two classes).*

Then in [FuS] they identified the central extension of $G_{mark} \cong T$ induced by ρ^{CF}:

Theorem 4.8 ([FuS]). *The central extension of G_{mark} induced by the almost G_{mark}-homomorphism ρ^{CF} (see §3.1 and §3.2 for ρ^{CF}) via the above described process, denoted by $\widehat{G}^{CF}_{mark}$, has the same presentation as $T_{1,0,0,0}$ (appearing in Thm 4.7):*

$$\widehat{G}^{CF}_{mark} \cong T_{1,0,0,0}, \tag{4.11}$$

(and this isomorphism is explicitly constructed via the above process) and hence Thm 4.7 corresponds to the extension class

$$c_{CF} = 12\chi \in H^2(G_{mark}). \tag{4.12}$$

We shall prove the following, which is our main result of the present paper:

Theorem 4.9. *The central extension of G_{mark} induced by the almost G_{mark}-homomorphism ρ^{Kash} (see §3.3) via the above described process, denoted by $\widehat{G}^{Kash}_{mark}$, has the same presentation as $T_{3,2,0,0}$ (appearing in Thm 4.7):*

$$\widehat{G}^{Kash}_{mark} \cong T_{3,2,0,0}, \tag{4.13}$$

(and this isomorphism is explicitly constructed via the above process) and hence by Thm 4.7 corresponds to the extension class

$$c_{Kash} = 6\chi \in H^2(G_{mark}). \tag{4.14}$$

As manifestly seen, the two G_{mark}-homomorphisms (almost linear presentations) ρ^{CF} and ρ^{Kash} yield central extensions of G_{mark} of distinct extension classes. Guo and Liu [GuLi] studied the relationship between the quantum coordinate change isomorphism Φ^q_g (3.23) for the Kashaev quantization and that for the Chekhov-Fock quantization. In (3.24) these isomorphisms are 'lifted' to the conjugation by $\rho(g)$, and it is only when we consider these lifts that we see this discrepancy between the two quantizations (on the induced central extension classes of the Ptolemy-Thompson group $G_{mark} \cong T$).

4.2. **Computation of the presentation of the central extension $\widehat{G}^{Kash}_{mark}$ of G_{mark} induced by the Kashaev representation ρ^{Kash}.** We give first an algebraic proof of Thm 4.9 here. We introduce the following notation first:

Definition 4.10. *For convenience, denote Kashaev's operators by*

$$\mathbf{A}_{[j]} := \rho(A_{[j]}), \quad \mathbf{T}_{[j][k]} := \rho(T_{[j][k]}), \quad \mathbf{P}_{(jk)} := \rho(P_{(jk)}) \tag{4.15}$$

the unitary operators on $\mathscr{M} = L^2(\mathbb{R}^{\mathbb{Q}^\times})$ (or sometimes we consider a dense subspace

$$\bigotimes_{j\in\mathbb{Q}^\times} M_j \subset \mathscr{M}, \quad \textit{where} \quad M_j \equiv L^2(\mathbb{R}, dx_j) \tag{4.16}$$

in some sense) corresponding to the generators of the group G_{dot} under (the almost linear representation) ρ as in Thm 3.10. Again for convenience, we follow (3.37), (3.38) *and denote*

$$\widehat{\alpha} := \rho^{Kash}(\alpha) = \mathbf{A}_{[-1]}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}\mathbf{P}_{\gamma_\alpha}, \tag{4.17}$$

$$\widehat{\beta} := \rho^{Kash}(\beta) = \mathbf{A}_{[-1]}\mathbf{P}_{\gamma_\beta}, \tag{4.18}$$

where for any permutation γ of $\mathbb{Q}^\times$ the operator $\mathbf{P}_\gamma$ permutes the factors according to the permutation γ:

$$(\mathbf{P}_\gamma f)(\{x_j\}_{j\in\mathbb{Q}^\times}) = f(\{x_{\gamma^{-1}(j)}\}_{j\in\mathbb{Q}^\times}), \quad f(\ldots, x_j, \ldots) \in \mathscr{M} = L^2(\mathbb{R}^{\mathbb{Q}^\times}). \tag{4.19}$$

Remark 4.11. *In the present paper, we avoid the discussion of what the infinite tensor product in* (4.16) *precisely means.*

Our task is now to take each of the relations of α, β for G_{mark} as in (2.9), and compute the lifted relation for the operators $\widehat{\alpha}$, $\widehat{\beta}$ on $\mathscr{M}$ coming from the almost G_{mark}-homomorphism ρ^{Kash} (i.e. Kashaev's almost linear representation).

We will be using the result of Prop. 3.11, which we can rewrite as follows for convenience:

$$\begin{aligned} &\mathbf{A}^3_{[j]} = id, \quad \mathbf{T}_{[k][\ell]}\mathbf{T}_{[j][k]} = \mathbf{T}_{[j][k]}\mathbf{T}_{[j][\ell]}\mathbf{T}_{[k][\ell]}, \\ &\mathbf{A}_{[j]}\mathbf{T}_{[j][k]}\mathbf{A}_{[k]} = \mathbf{A}_{[k]}\mathbf{T}_{[k][j]}\mathbf{A}_{[j]}, \quad \mathbf{T}_{[j][k]}\mathbf{A}_{[j]}\mathbf{T}_{[k][j]} = \zeta\mathbf{A}_{[j]}\mathbf{A}_{[k]}\mathbf{P}_{(jk)}, \end{aligned} \tag{4.20}$$

where $j, k, \ell \in \mathbb{Q}^\times$ are mutually distinct, ζ is as in (3.34) (a complex number of modulus one, which is not a root of unity when b^2 is irrational). The following variants of (4.20) will come handy:

$$\begin{aligned} &\mathbf{T}^{-1}_{[j][k]}\mathbf{A}_{[j]}\mathbf{A}_{[k]} = \zeta^{-1}\mathbf{A}_{[j]}\mathbf{P}_{(jk)}\mathbf{T}_{[j][k]}, \quad \mathbf{T}^{-1}_{[j][k]}\mathbf{A}^2_{[j]}\mathbf{A}_{[k]} = \mathbf{A}^2_{[j]}\mathbf{A}_{[k]}\mathbf{T}^{-1}_{[k][j]}, \\ &\mathbf{T}^{-1}_{[j][k]}\mathbf{T}^{-1}_{[k][\ell]} = \mathbf{T}^{-1}_{[k][\ell]}\mathbf{T}^{-1}_{[j][\ell]}\mathbf{T}^{-1}_{[j][k]}, \quad \mathbf{T}^{-1}_{[k][j]}\mathbf{A}^2_{[j]}\mathbf{T}^{-1}_{[j][k]} = \zeta^{-1}\mathbf{P}_{(jk)}\mathbf{A}^2_{[j]}\mathbf{A}^2_{[k]}. \end{aligned} \tag{4.21}$$

We'll also use the following relations for the index permutations

$$\mathbf{P}^2_{(jk)} = id, \quad \mathbf{P}_{(jk)}\mathbf{f}_{\ldots,[j],\ldots,[k],\ldots}\mathbf{P}_{(jk)} = \mathbf{f}_{\ldots,[k],\ldots,[j],\ldots}, \quad \mathbf{P}_{(jk)} = \mathbf{P}_{(kj)}, \tag{4.22}$$

where $\mathbf{f}_{\ldots,[j],\ldots,[k]\ldots}$ is any word in $\mathbf{A}_{[\cdot]}$, $\mathbf{T}_{[\cdot][\cdot]}$, $\mathbf{P}_{(\cdot\,\cdot)}$ (conjugation by $\mathbf{P}_{(jk)}$ results in exchanging the subscripts $[j]$ and $[k]$), as well as the fact that any two expressions whose collections of subscript indices don't intersect with each other commute.

We begin from the easier relations:

Proposition 4.12. *The operators $\widehat{\alpha}$* (4.17) *and $\widehat{\beta}$* (4.18) *satisfy*

$$\widehat{\beta}^3 = 1 \quad \textit{and} \quad \widehat{\alpha}^4 = \zeta^{-2}. \tag{4.23}$$

Proof. Recall that γ_β fixes -1, therefore $\mathbf{P}_{\gamma_\beta}$ commutes with $\mathbf{A}_{[-1]}$; from (4.18),

$$\widehat{\beta}^3 = (\mathbf{A}_{[-1]}\mathbf{P}_{\gamma_\beta})^3 = \mathbf{A}^3_{[-1]}\mathbf{P}^3_{\gamma_\beta}. \tag{4.24}$$

From (4.20) we know $\mathbf{A}^3_{[-1]} = id$. From (4.19) it's easy to see

$$\mathbf{P}_{\gamma_1\circ\gamma_2} = \mathbf{P}_{\gamma_1}\mathbf{P}_{\gamma_2} \tag{4.25}$$

for any two permutations γ_1, γ_2 of $\mathbb{Q}^\times$. Thus we have $\mathbf{P}^3_{\gamma_\beta} = \mathbf{P}_{\gamma^3_\beta}$. By applying the definition of γ_β (as in Def. 2.34) three times, it is easy to see that γ^3_β is the identity permutation of $\mathbb{Q}^\times$. Hence $\mathbf{P}_{\gamma^3_\beta} = id$. This yields $\widehat{\beta}^3 = 1$.

Take (4.17) and square it. Since γ_α fixes -1 and 1, we know that $\mathbf{P}_{\gamma_\alpha}$ commutes with $\mathbf{A}_{[-1]}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}$;

$$\begin{aligned}
\widehat{\alpha}^2 &= (\mathbf{A}_{[-1]}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}\mathbf{P}_{\gamma_\alpha})^2 = \mathbf{A}_{[-1]}\mathbf{T}^{-1}_{[-1][1]}\underline{\mathbf{A}_{[1]}\mathbf{A}_{[-1]}}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}\mathbf{P}^2_{\gamma_\alpha} \\
&= \mathbf{A}_{[-1]}\underline{\mathbf{T}^{-1}_{[-1][1]}(\mathbf{A}_{[-1]}\mathbf{A}_{[1]})}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}\mathbf{P}^2_{\gamma_\alpha} \quad (\because \mathbf{A}_{[-1]} \text{ and } \mathbf{A}_{[1]} \text{ commute}) \\
&\overset{(4.21)}{=} \mathbf{A}_{[-1]}(\zeta^{-1}\mathbf{A}_{[-1]}\underline{\mathbf{P}_{(-1\,1)}\cancel{\mathbf{T}_{[-1][1]}})\cancel{\mathbf{T}^{-1}_{[-1][1]}}\mathbf{A}_{[1]}}\mathbf{P}^2_{\gamma_\alpha}
\end{aligned} \tag{4.26}$$

$$\overset{(4.22)}{=} \zeta^{-1}\underline{\mathbf{A}^3_{[-1]}}\mathbf{P}_{(-1\,1)}\mathbf{P}^2_{\gamma_\alpha} \overset{(4.20)}{=} \zeta^{-1}\,\mathbf{P}_{(-1\,1)}\mathbf{P}^2_{\gamma_\alpha}, \tag{4.27}$$

where we underlined the part which is being replaced in each step. Since γ_α fixes -1 and 1, we know $\mathbf{P}_{\gamma_\alpha}$ commutes with $\mathbf{P}_{(-1\,1)}$, and therefore

$$\widehat{\alpha}^4 = (\widehat{\alpha}^2)^2 = (\zeta^{-1}\,\mathbf{P}_{(-1\,1)}\mathbf{P}^2_{\gamma_\alpha})^2 = \zeta^{-2}\,\underline{\mathbf{P}^2_{(-1\,1)}}\,\underline{\mathbf{P}^4_{\gamma_\alpha}} \overset{(4.22),(4.25)}{=} \zeta^{-2}\,\mathbf{P}_{\gamma^4_\alpha}. \tag{4.28}$$

By applying the definition of γ_α (as in Def. 2.34) four times, it is easy to see that γ^4_α is the identity permutation of $\mathbb{Q}^\times$. Hence $\mathbf{P}^4_{\gamma_\alpha} = id$. This yields $\widehat{\alpha}^4 = \zeta^{-2}$. □

From now on, the trivial step of switching the order of some factors as we did in (4.26) may not be explicitly shown. Proof of the following result is a little bit more involved.

Proposition 4.13. *The operators* $\widehat{\alpha}$ (4.17) *and* $\widehat{\beta}$ (4.18) *satisfy*

$$(\widehat{\beta}\widehat{\alpha})^5 = \zeta^{-3}. \tag{4.29}$$

Proof. We first observe that (4.19) implies

$$\mathbf{P}_\gamma(\cdots\mathbf{A}_{[j]}\cdots\mathbf{T}_{[k][\ell]}\cdots\mathbf{P}_{(nm)}\cdots) = (\cdots\mathbf{A}_{[\gamma(j)]}\cdots\mathbf{T}_{[\gamma(k)][\gamma(\ell)]}\cdots\mathbf{P}_{(\gamma(n)\gamma(m))}\cdots)\mathbf{P}_\gamma, \tag{4.30}$$

for any permutation γ of $\mathbb{Q}^\times$. The middle equation of (4.22) is a special case of (4.30). Now we take (4.17) and (4.18):

$$\begin{aligned}
\widehat{\beta}\widehat{\alpha} = (\mathbf{A}_{[-1]}\underline{\mathbf{P}_{\gamma_\beta})(\mathbf{A}_{[-1]}\mathbf{T}^{-1}_{[-1][1]}\mathbf{A}_{[1]}}\mathbf{P}_{\gamma_\alpha}) &\overset{(4.30)}{=} \mathbf{A}_{[-1]}\mathbf{A}_{[\gamma_\beta(-1)]}\mathbf{T}^{-1}_{[\gamma_\beta(-1)][\gamma_\beta(1)]}\mathbf{A}_{[\gamma_\beta(1)]}\underline{\mathbf{P}_{\gamma_\beta}\mathbf{P}_{\gamma_\alpha}} \\
&\overset{(2.25),(4.25)}{=} \mathbf{A}^2_{[-1]}\mathbf{T}^{-1}_{[-1][-\frac{1}{2}]}\mathbf{A}_{[-\frac{1}{2}]}\mathbf{P}_{\gamma_\beta\circ\gamma_\alpha}.
\end{aligned}$$

From (2.24) and (2.25) we get $\gamma_\beta\circ\gamma_\alpha : -1 \mapsto -1$, $1 \to -\frac{1}{2}$, $-\frac{1}{2} \mapsto 1$, hence $\gamma_\beta\circ\gamma_\alpha$ leaves $\{-1, -\frac{1}{2}, 1\}$ invariant, and the action on this set is just exchanging 1 and $-\frac{1}{2}$. Therefore we can write

$$\gamma_\beta\circ\gamma_\alpha = (-\frac{1}{2}\,1)\circ\gamma \tag{4.31}$$

for some permutation γ of $\mathbb{Q}^\times$ which fixes $-1, -\frac{1}{2}, 1$. Now using (4.25) we write $\mathbf{P}_{\gamma_\beta \circ \gamma_\alpha} = \mathbf{P}_{(-\frac{1}{2}\,1)} \mathbf{P}_\gamma$. Since γ fixes $-1, -\frac{1}{2}, 1$, we know $\mathbf{P}_\gamma$ commutes with the expression $\mathbf{A}_{[-1]}^2 \mathbf{T}_{[-1][-\frac{1}{2}]}^{-1} \mathbf{A}_{[-\frac{1}{2}]} \mathbf{P}_{(-\frac{1}{2}\,1)}$. We thus denote this expression by

$$(\widehat{\beta}\widehat{\alpha})_0 := \mathbf{A}_{[-1]}^2 \mathbf{T}_{[-1][-\frac{1}{2}]}^{-1} \mathbf{A}_{[-\frac{1}{2}]} \mathbf{P}_{(-\frac{1}{2}\,1)}, \tag{4.32}$$

so that

$$\widehat{\beta}\widehat{\alpha} = (\widehat{\beta}\widehat{\alpha})_0 \, \mathbf{P}_\gamma = \mathbf{P}_\gamma \, (\widehat{\beta}\widehat{\alpha})_0. \tag{4.33}$$

Hence we would have

$$(\widehat{\beta}\widehat{\alpha})^5 = (\widehat{\beta}\widehat{\alpha})_0^5 \, \mathbf{P}_\gamma^5 = (\widehat{\beta}\widehat{\alpha})_0^5 \, \mathbf{P}_{\gamma^5}. \tag{4.34}$$

Think of applying the move $(\beta\alpha)^5$ to the standard marked tessellation τ^*_{mark} (ten moves in total). By drawing the picture for each step, we can observe that all ideal triangles of τ^*_{mark} (here ideal triangles are viewed as subsets of $\mathbb{D}$) remain intact during this whole process of ten moves, except the three which are labeled by $-1, -\frac{1}{2}, 1$ according to the labeling rule L^* of $F(\tau^*_{mark}) = \tau^*_{dot} = (\tau^*, D^*, L^*)$. And we know that $(\beta\alpha)^5$ is the identity move on $\mathcal{F}tess_{mark}$. Now, by following definition of γ_α, γ_β (see Def. 2.34) and γ (see (4.31)), we can deduce that γ^5 is the identity permutation of $\mathbb{Q}^\times$, thus $\mathbf{P}_{\gamma^5} = 1$.

Thus it remains to prove $(\widehat{\beta}\widehat{\alpha})_0^5 = \zeta^{-3}$. From its definition (4.32), $(\widehat{\beta}\widehat{\alpha})_0$ can be thought of as an operator on $M_{-1} \otimes M_{-\frac{1}{2}} \otimes M_1$ (see (4.16) for notation M_j). For the ease of notation, we replace the subscripts $[-1], [-\frac{1}{2}], [1]$ with $1, 2, 3$ respectively, where now the subscript $j \in \{1, 2, 3\}$ means the j-th factor of $M_{-1} \otimes M_{-\frac{1}{2}} \otimes M_1$. For example, $\mathbf{A}_{[-1]}$ will now be denoted by $\mathbf{A}_1$, and $\mathbf{T}_{[-1][-\frac{1}{2}]}$ by $\mathbf{T}_{12}$, indicating on which factor the operators are acting on. The permutation operators will be denoted without the parentheses, e.g. $\mathbf{P}_{(-\frac{1}{2}\,1)}$ will be denoted by $\mathbf{P}_{23}$. Then we now can rewrite (4.32) as:

$$(\widehat{\beta}\widehat{\alpha})_0 = \mathbf{A}_1^2 \mathbf{T}_{12}^{-1} \mathbf{A}_2 \mathbf{P}_{23} : M_{-1} \otimes M_{-\frac{1}{2}} \otimes M_1 \longrightarrow M_{-1} \otimes M_{-\frac{1}{2}} \otimes M_1. \tag{4.35}$$

We first note that

$$(\widehat{\beta}\widehat{\alpha})_0^2 = (\mathbf{A}_1^2 \mathbf{T}_{12}^{-1} \mathbf{A}_2 \underline{\mathbf{P}_{23})(\mathbf{A}_1^2 \mathbf{T}_{12}^{-1} \mathbf{A}_2 \mathbf{P}_{23}}) \overset{(4.22)}{=} \mathbf{A}_1^2 \underline{\mathbf{T}_{12}^{-1} \mathbf{A}_2 \mathbf{A}_1^2} \mathbf{T}_{13}^{-1} \mathbf{A}_3 \tag{4.36}$$

$$\overset{(4.21)}{=} \underline{\mathbf{A}_1^2 \mathbf{A}_1^2} \mathbf{A}_2 \mathbf{T}_{21}^{-1} \mathbf{T}_{13}^{-1} \mathbf{A}_3 \overset{(4.20)}{=} \mathbf{A}_1 \mathbf{A}_2 \mathbf{T}_{21}^{-1} \mathbf{T}_{13}^{-1} \mathbf{A}_3. \tag{4.37}$$

Putting together (4.35) and (4.37), we get

$$
\begin{aligned}
(\widehat{\beta}\widehat{\alpha})_0^5 &= (\widehat{\beta}\widehat{\alpha})_0^2(\widehat{\beta}\widehat{\alpha})_0(\widehat{\beta}\widehat{\alpha})_0^2 = (\underline{\mathbf{A}_1\mathbf{A}_2\mathbf{T}_{21}^{-1}\mathbf{T}_{13}^{-1}\mathbf{A}_3)(\mathbf{A}_1^2\mathbf{T}_{12}^{-1}\mathbf{A}_2\mathbf{P}_{23}})(\mathbf{A}_1\mathbf{A}_2\mathbf{T}_{21}^{-1}\mathbf{T}_{13}^{-1}\mathbf{A}_3) \\
&\overset{(4.22)}{=} (\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}\mathbf{T}_{12}^{-1}\mathbf{A}_2\mathbf{A}_1^2\underline{\mathbf{T}_{13}^{-1}\mathbf{A}_3)\mathbf{A}_1}\mathbf{A}_2\underline{\mathbf{T}_{21}^{-1}\mathbf{T}_{13}^{-1}}\mathbf{A}_3 \\
&\overset{(4.21)}{=} \mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}\mathbf{T}_{12}^{-1}\mathbf{A}_2\mathbf{A}_1^2(\zeta^{-1}\mathbf{A}_1\,\underbrace{\mathbf{P}_{(13)}\underline{\cancel{\mathbf{T}_{13}})\mathbf{A}_2(\cancel{\mathbf{T}_{13}^{-1}}}\mathbf{T}_{23}^{-1}\mathbf{T}_{21}^{-1})\mathbf{A}_3} \\
&\overset{(4.22)}{=} \zeta^{-1}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}\mathbf{T}_{12}^{-1}\underline{\mathbf{A}_2\cancel{\mathbf{A}_1^2}\cancel{\mathbf{A}_1}(\mathbf{A}_2}\mathbf{T}_{21}^{-1}\mathbf{T}_{23}^{-1}\mathbf{A}_1\mathbf{P}_{(13)}) \\
&\overset{(4.20)}{=} \zeta^{-1}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}\underline{\mathbf{T}_{12}^{-1}(\mathbf{A}_2^2)\mathbf{T}_{21}^{-1}}\mathbf{T}_{23}^{-1}\mathbf{A}_1\mathbf{P}_{(13)} \\
&\overset{(4.21)}{=} \zeta^{-1}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}(\zeta^{-1}\underline{\mathbf{P}_{(21)}\mathbf{A}_2^2\mathbf{A}_1^2)\mathbf{T}_{23}^{-1}\mathbf{A}_1}\mathbf{P}_{(13)} \\
&\overset{(4.22)}{=} \zeta^{-2}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\mathbf{T}_{31}^{-1}(\mathbf{A}_1^2\underline{\mathbf{A}_2^2\mathbf{T}_{13}^{-1}}\mathbf{A}_2\mathbf{P}_{(12)})\mathbf{P}_{(13)} \\
&= \zeta^{-2}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3\underline{\mathbf{T}_{31}^{-1}\mathbf{A}_1^2(\mathbf{T}_{13}^{-1}}\ \underline{\cancel{\mathbf{A}_2^2})\cancel{\mathbf{A}_2}}\mathbf{P}_{(12)}\mathbf{P}_{(13)} \\
&\overset{(4.20),(4.21)}{=} \zeta^{-2}\mathbf{P}_{23}\mathbf{A}_1\mathbf{A}_3(\zeta^{-1}\underline{\mathbf{P}_{13}\mathbf{A}_1^2\mathbf{A}_3^2)\mathbf{P}_{(12)}\mathbf{P}_{(13)}} \\
&\overset{(4.22)}{=} \zeta^{-3}\mathbf{P}_{23}\underbrace{\cancel{\mathbf{A}_1}\underline{\cancel{\mathbf{A}_3}(\cancel{\mathbf{A}_3^2}}\cancel{\mathbf{A}_1^2}}\mathbf{P}_{(32)}) \overset{(4.20)}{=} \zeta^{-3}\underline{\cancel{\mathbf{P}_{23}\mathbf{P}_{(32)}}} \overset{(4.22)}{=} \zeta^{-3}.
\end{aligned}
$$

□

A key observation in the above proof is that $(\widehat{\beta}\widehat{\alpha})^5$ acts in an interesting way (i.e. involving $\mathbf{A}_\cdot$'s and $\mathbf{T}_{\cdot,\cdot}$'s) only on the three factors $M_{-1}, M_{-\frac{1}{2}}, M_1$, and acts as a permutation on the other factors, where this permutation is in fact the identity permutation. Similarly, when checking the remaining two relations, the relevant operators will act in an interesting way only on a few factors, and act as a certain permutation on the other factors. Hence we'll focus on those few factors as we've done above.

Proposition 4.14. *The operators* $\widehat{\alpha}$ (4.17) *and* $\widehat{\beta}$ (4.18) *satisfy*

$$(\widehat{\beta}\widehat{\alpha}\widehat{\beta})(\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2) = (\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2)(\widehat{\beta}\widehat{\alpha}\widehat{\beta}), \tag{4.38}$$

or equivalently,

$$[\widehat{\beta}\widehat{\alpha}\widehat{\beta},\ \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2] = 1. \tag{4.39}$$

Proof. We first study the LHS of (4.38), while we now ignore the hats (so think of it as a word in α, β rather than $\widehat{\alpha}$, $\widehat{\beta}$) and view it as acting on $\mathcal{F}tess_{mark}$; thus we read from right. We keep track of how each step (first step being α^2, then β, then α, etc) transforms marked tessellations; this is recorded in Fig. 12. For convenience of drawing, we use the standard marked tessellation τ^*_{mark} as the starting point.

As can be deduced from Fig. 12, the operator $(\widehat{\beta}\widehat{\alpha}\widehat{\beta})(\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2)$ acts in an interesting way (i.e. involving $\mathbf{A}_\cdot$ and $\mathbf{T}_{\cdot,\cdot}$) only on the factors $M_{-2}, M_{-1}, M_{\frac{1}{2}}, M_1$, and acts as a permutation on the other factors. Since the element of $\mathcal{F}tess_{mark}$ that we end up with at the end of Fig. 12 is same as that of Fig. 13 (i.e. from the RHS of (4.38)), we know that the permutation action on the other factors besides the above four is the same for the LHS and the RHS of (4.38).

Therefore we indeed can focus on the action on $M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1$. However, as we apply the LHS of (4.38), the four factors involved get replaced at each step, as can be seen in Fig.

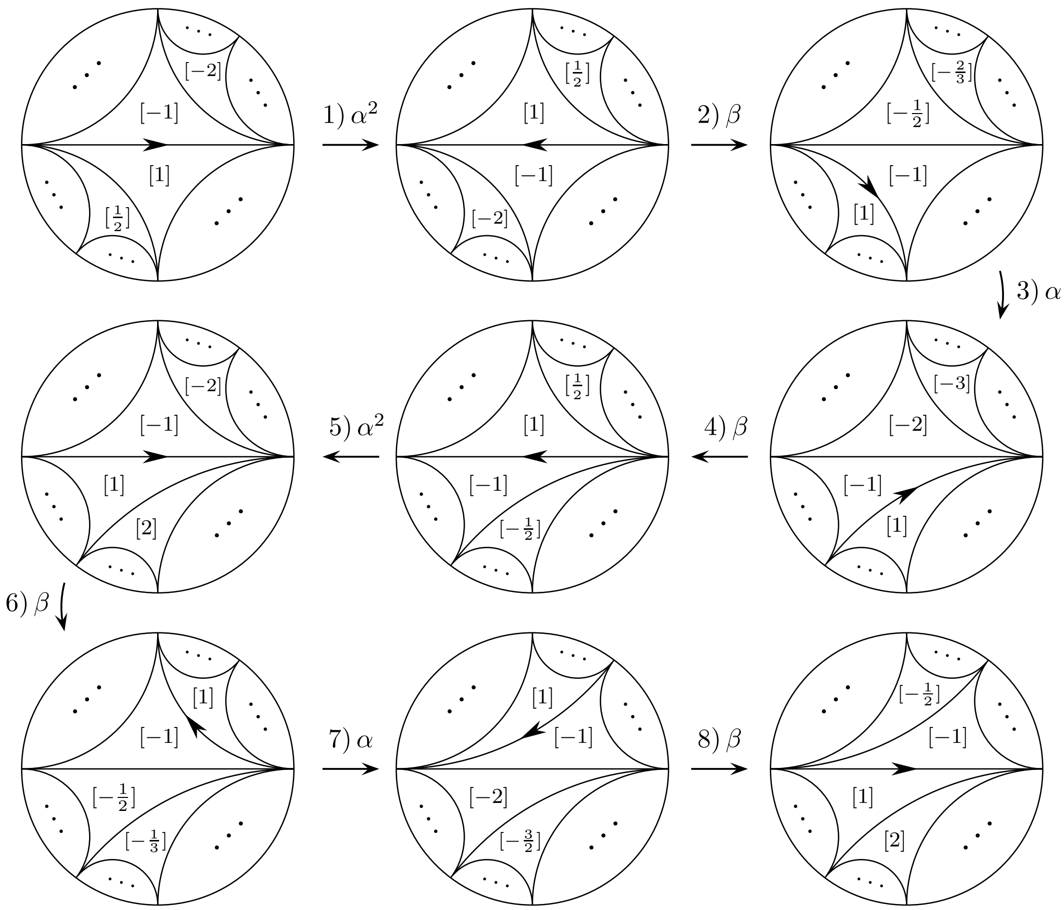


FIGURE 12. The LHS of (4.38), the action of $\beta\alpha\beta\,\alpha^2\beta\alpha\beta\alpha^2$ on τ^*_{mark}

12. In order to use the notation for the subscript indices of $\mathbf{A}_\cdot$ and $\mathbf{T}_{\cdot,\cdot}$ as done in the proof of Prop. 4.13 (i.e. instead of $\mathbb{Q}^\times$-labels for triangles (written with brackets $[\cdot]$), we just indicate the position of the relevant factors; e.g. $\mathbf{T}_{34}$ on $M_{-2} \otimes M_{-1} \otimes M_{\frac{1}{2}} \otimes M_1$ would stand for $\mathbf{T}_{[\frac{1}{2}][1]}$ in the bracket notation), it is necessary to keep track of the four involved factors at each stage. This is the reason why it's necessary to have Fig. 12.

We break up the LHS of (4.38) from the right into eight steps, namely, 1) $\widehat{\alpha}^2$, 2) $\widehat{\beta}$, 3) $\widehat{\alpha}$, 4) $\widehat{\beta}$, 5) $\widehat{\alpha}^2$, 6) $\widehat{\beta}$, 7) $\widehat{\alpha}$, 8) $\widehat{\beta}$. Following the definitions (4.17), (4.18) of $\widehat{\alpha}, \widehat{\beta}$, together with the result (4.27) $\widehat{\alpha}^2 = \zeta^{-1}\mathbf{P}_{(-1\,1)}\mathbf{P}^2_{\gamma_\alpha}$ (therefore $\widehat{\alpha}^2$ acts on $M_{-1} \otimes M_1 \to M_{-1} \otimes M_1$ as $\zeta^{-1}\mathbf{P}_{12}$ and as permutation on the other factors according to the picture for the α^2 action on marked tessellations), and keeping track of which factor gets permuted to which factor (i.e. which triangle is matched with which triangle at each step in the picture, which is part of the definitions of $\widehat{\alpha}, \widehat{\beta}$), we complete the following list of the operators for each stage of the LHS of (4.38) (we order the tensor product at each stage so that the $\mathbb{Q}^\times$-subscripts are in the increasing

order):

$$
\begin{array}{rcll}
M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1 & \xrightarrow{1)\,\widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1 & : \mathbf{P}_{13}(\zeta^{-1}\mathbf{P}_{24}) \\
M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1 & \xrightarrow{2)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2 \\
M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_1 & \xrightarrow{3)\,\widehat{\alpha}} & M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4) \\
M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & \xrightarrow{4)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_1 & : \mathbf{P}_{24}\mathbf{P}_{13}\mathbf{A}_3 \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_1 & \xrightarrow{5)\,\widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_1\otimes M_2 & : \mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14}) \\
M_{-2}\otimes M_{-1}\otimes M_1\otimes M_2 & \xrightarrow{6)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2 \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_1 & \xrightarrow{7)\,\widehat{\alpha}} & M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4) \\
M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1 & \xrightarrow{8)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_2 & : \mathbf{P}_{24}\mathbf{P}_{13}\mathbf{A}_3.
\end{array}
$$

Therefore the LHS of (4.38) is

$$
\begin{aligned}
&(\mathbf{P}_{24}\underbrace{\underline{\mathbf{P}_{13}}\mathbf{A}_3)(\mathbf{P}_{12}\underline{\mathbf{P}_{13}}}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4))(\mathbf{P}_{23}\mathbf{P}_{24}\underline{\mathbf{P}_{12}\mathbf{A}_2)(\mathbf{P}_{14}\mathbf{P}_{12}}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14}))\\
&\cdot(\mathbf{P}_{24}\underline{\mathbf{P}_{13}\mathbf{A}_3)(\mathbf{P}_{12}\mathbf{P}_{13}}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4))(\mathbf{P}_{23}\underline{\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2)(\mathbf{P}_{13}(\zeta^{-1}\mathbf{P}_{24}})) \\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{24}(\mathbf{A}_1\underline{\cancel{\mathbf{P}_{32}})\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4\cancel{\mathbf{P}_{23}}}\ \underline{\cancel{\mathbf{P}_{24}}(\mathbf{A}_1\cancel{\mathbf{P}_{24}}})\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{P}_{24}(\mathbf{A}_1\underline{\cancel{\mathbf{P}_{32}})\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4\cancel{\mathbf{P}_{23}}}(\mathbf{P}_{14}\mathbf{A}_4) \\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{24}\underline{\underline{\mathbf{A}_1(\mathbf{A}_1}\mathbf{T}_{14}^{-1}\mathbf{A}_4)(\mathbf{A}_1)\mathbf{P}_{34}}\ \underline{\mathbf{P}_{14}\mathbf{P}_{24}\underline{\mathbf{A}_1(\mathbf{A}_1}\mathbf{T}_{14}^{-1}\mathbf{A}_4)\mathbf{P}_{14}}\mathbf{A}_4\mathbf{P}_{13} \\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{24}(\mathbf{P}_{34}\underline{\mathbf{A}_1^2\mathbf{T}_{13}^{-1}\mathbf{A}_3\mathbf{A}_1)(\mathbf{P}_{21}}\mathbf{A}_4^2\mathbf{T}_{41}^{-1}\mathbf{A}_1)\mathbf{A}_4\mathbf{P}_{13} \\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{24}\mathbf{P}_{34}(\mathbf{P}_{21}\underline{\mathbf{A}_2^2\mathbf{T}_{23}^{-1}\mathbf{A}_3\mathbf{A}_2)\mathbf{A}_4^2\mathbf{T}_{41}^{-1}\mathbf{A}_1\mathbf{A}_4\mathbf{P}_{13}} \\
&\overset{(4.22)}{=}\zeta^{-2}\underline{\mathbf{P}_{24}\mathbf{P}_{34}\mathbf{P}_{21}(\mathbf{P}_{13}}\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4) \\
&=\zeta^{-2}\mathbf{P}_{34}\mathbf{P}_{12}\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4,
\end{aligned}
$$

where the last line is an easy exercise about permutations.

We do likewise for the RHS of (4.38); we break up the RHS of (4.38) from the right into eight steps: 1) $\widehat{\beta}$, 2) $\widehat{\alpha}$, 3) $\widehat{\beta}$, 4) $\widehat{\alpha}^2$, 5) $\widehat{\beta}$, 6) $\widehat{\alpha}$, 7) $\widehat{\beta}$, 8) $\widehat{\alpha}^2$. If we ignore the hats, i.e. if we think of $\widehat{\alpha},\widehat{\beta}$ as just α,β, the action on the marked tessellations, starting from the standard one τ^*_{mark}, is depicted in Fig. 13.

As in the LHS case, the RHS operator $(\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2)(\widehat{\beta}\widehat{\alpha}\widehat{\beta})$ acts in an interesting way only on the factors $M_{-2},M_{-1},M_{\frac{1}{2}},M_1$. Just as before, we complete the following list of the operators for each stage of RHS of (4.38) (again we order the tensor product at each stage so that the

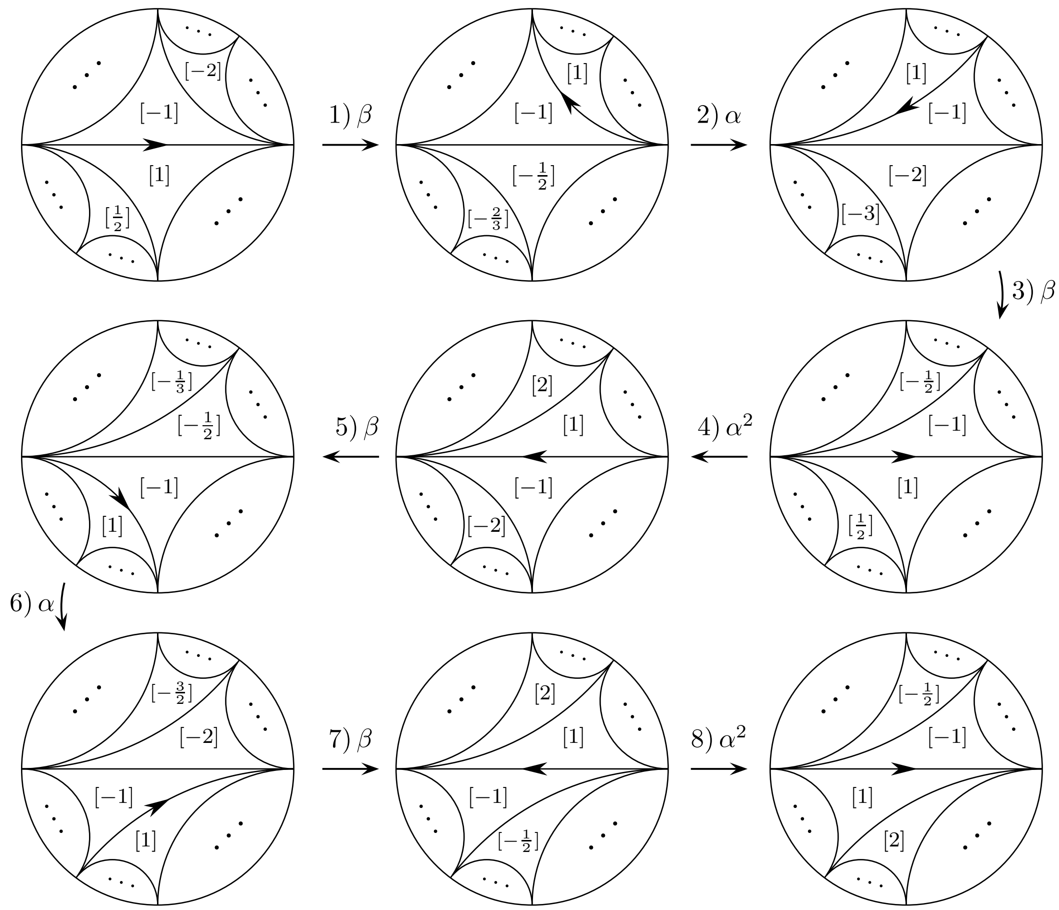


FIGURE 13. The RHS of (4.38), the action of $\alpha^2\beta\alpha\beta\alpha^2\,\beta\alpha\beta$ on τ^*_{mark}

$\mathbb{Q}^\times$-subscripts are in the increasing order):

$$
\begin{array}{rcll}
M_{-2}\otimes M_{-1}\otimes M_{\frac12}\otimes M_1 & \xrightarrow{1)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac23}\otimes M_{-\frac12}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2\\
M_{-1}\otimes M_{-\frac23}\otimes M_{-\frac12}\otimes M_1 & \xrightarrow{2)\,\widehat{\alpha}} & M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4)\\
M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & \xrightarrow{3)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac12}\otimes M_{\frac12}\otimes M_1 & : \mathbf{P}_{24}\mathbf{P}_{13}\mathbf{A}_3\\
M_{-1}\otimes M_{-\frac12}\otimes M_{\frac12}\otimes M_1 & \xrightarrow{4)\,\widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_1\otimes M_2 & : \mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14})\\
M_{-2}\otimes M_{-1}\otimes M_1\otimes M_2 & \xrightarrow{5)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac12}\otimes M_{-\frac13}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2\\
M_{-1}\otimes M_{-\frac12}\otimes M_{-\frac13}\otimes M_1 & \xrightarrow{6)\,\widehat{\alpha}} & M_{-2}\otimes M_{-\frac32}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4)\\
M_{-2}\otimes M_{-\frac32}\otimes M_{-1}\otimes M_1 & \xrightarrow{7)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac12}\otimes M_1\otimes M_2 & : \mathbf{P}_{24}\mathbf{P}_{13}\mathbf{A}_3\\
M_{-1}\otimes M_{-\frac12}\otimes M_1\otimes M_2 & \xrightarrow{8)\,\widehat{\alpha}^2} & M_{-1}\otimes M_{-\frac12}\otimes M_1\otimes M_2 & : \mathbf{P}_{24}(\zeta^{-1}\mathbf{P}_{13}).
\end{array}
$$

Therefore the RHS of (4.38) is

$$
\begin{aligned}
&(\mathbf{P}_{24}(\zeta^{-1}\underbrace{\underline{\cancel{\mathbf{P}_{13}}}))(\mathbf{P}_{24}\underline{\cancel{\mathbf{P}_{13}}}}\mathbf{A}_3)(\underline{\mathbf{P}_{12}\mathbf{P}_{13}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4))(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}}\mathbf{A}_2)\\
&\cdot(\underline{\mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14}}))(\mathbf{P}_{24}\underline{\mathbf{P}_{13}\mathbf{A}_3)(\mathbf{P}_{12}\mathbf{P}_{13}}(\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4))(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2)\\
&\overset{(4.22)}{=}\zeta^{-2}\underline{\cancel{\mathbf{P}_{24}}(\cancel{\mathbf{P}_{24}})}\underline{\mathbf{A}_3(\mathbf{P}_{23}}\mathbf{A}_2\mathbf{T}_{24}^{-1}\mathbf{A}_4\underline{\mathbf{P}_{13}\mathbf{P}_{14})\mathbf{A}_2(\mathbf{P}_{42}\mathbf{P}_{31}})\mathbf{P}_{24}(\mathbf{A}_1\underline{\cancel{\mathbf{P}_{32}})\mathbf{A}_1\mathbf{T}_{14}^{-1}\mathbf{A}_4\cancel{\mathbf{P}_{23}}}\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-2}(\underline{\mathbf{P}_{23}\underline{\mathbf{A}_2)\mathbf{A}_2}\mathbf{T}_{24}^{-1}\mathbf{A}_4(\mathbf{P}_{34}}\mathbf{A}_2\underline{\cancel{\mathbf{P}_{42}})\cancel{\mathbf{P}_{24}}}\ \underline{\mathbf{A}_1(\mathbf{A}_1}\mathbf{T}_{14}^{-1}\mathbf{A}_4)\mathbf{P}_{24}\mathbf{P}_{12}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-2}(\mathbf{P}_{34}\underline{\mathbf{P}_{24}\mathbf{A}_2^2\mathbf{T}_{23}^{-1}\mathbf{A}_3)\mathbf{A}_2\mathbf{A}_1^2\mathbf{T}_{14}^{-1}\mathbf{A}_4\mathbf{P}_{24}}\mathbf{P}_{12}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{34}(\underline{\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4\mathbf{A}_1^2\mathbf{T}_{12}^{-1}\mathbf{A}_2)\mathbf{P}_{12}}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-2}\mathbf{P}_{34}(\mathbf{P}_{12}\underbrace{\underline{\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4}\ \underline{\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1)\mathbf{A}_2}}\\
&=\zeta^{-2}\mathbf{P}_{34}\mathbf{P}_{12}(\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4),
\end{aligned}
$$

where the last equality holds because any two expressions whose collections of subscripts don't intersect commute. We can now see by inspection that the LHS and the RHS of (4.38) are both $\zeta^{-2}\mathbf{P}_{34}\mathbf{P}_{12}(\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_4^2\mathbf{T}_{43}^{-1}\mathbf{A}_3\mathbf{A}_4)$, and therefore identical. ☐

Note that in the above proof, we used only trivial permutation relations (4.22), and no non-trivial ones (4.20). We now go on to compute the last (lifted) relation:

Proposition 4.15. *The operators* $\widehat{\alpha}$ (4.17) *and* $\widehat{\beta}$ (4.18) *satisfy*

$$(\widehat{\beta}\widehat{\alpha}\widehat{\beta})(\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2) = (\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2)(\widehat{\beta}\widehat{\alpha}\widehat{\beta}), \tag{4.40}$$

or equivalently,

$$[\widehat{\beta}\widehat{\alpha}\widehat{\beta},\ \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2] = 1. \tag{4.41}$$

Proof. We proceed as in the proof of Prop. 4.14. We break up the LHS of (4.40) from the right into twelve steps: 1) $\widehat{\alpha}^2$, 2) $\widehat{\beta}^2$, 3) $\widehat{\alpha}^2$, 4) $\widehat{\beta}$, 5) $\widehat{\alpha}$, 6) $\widehat{\beta}$, 7) $\widehat{\alpha}^2$, 8) $\widehat{\beta}$, 9) $\widehat{\alpha}^2$, 10) $\widehat{\beta}$, 11) $\widehat{\alpha}$, 12) $\widehat{\beta}$. We ignore the hats of $\widehat{\alpha},\widehat{\beta}$, and view them as α,β acting on marked tessellations, and we record these actions, starting from the standard one τ^*_{mark}, in Fig. 14. As can be deduced from the picture, the LHS of (4.40) acts in an interesting way only on the factors $M_{-2}, M_{-1}, M_1, M_{\frac{3}{2}}, M_2$, and acts as a permutation on the other factors. Since the element of $\mathcal{F}tess_{mark}$ that we end up with after the action of the LHS on τ^*_{mark} is same as that for the RHS as can be seen in Figures 14 and 15, the permutation action on the 'non-interesting' factors is the same for the LHS and the RHS of (4.40).

Hence, as before, we can focus on the 'interesting' factors. As done in the proof of Prop. 4.14, we follow the twelve steps for the LHS as depicted in Fig. 14 and write down the corresponding operators in terms of $\mathbf{A}_\cdot, \mathbf{T}_{\cdot,\cdot}, \mathbf{P}_{\cdot\cdot}$ (subscripts are now without brackets, and they stand for the position of the relevant factor at each stage; see the proof of Prop. 4.13) on the relevant factors (again, we order the tensor product at each stage so that the $\mathbb{Q}^\times$-subscripts are in the

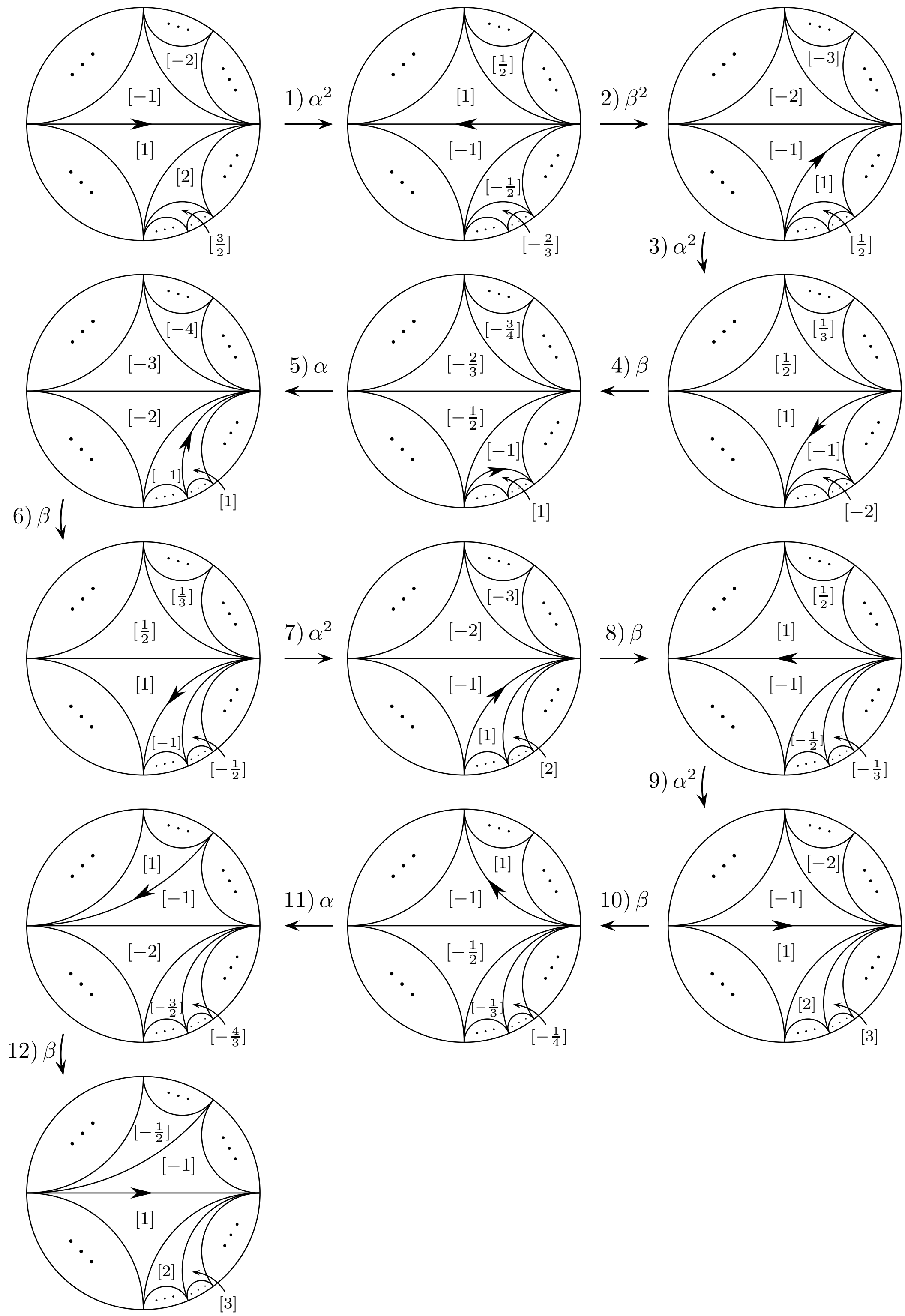


FIGURE 14. The LHS of (4.40), the action of $\beta\alpha\beta\,\alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2$ on τ^*_{mark}

increasing order):

$$
\begin{array}{rcll}
M_{-2}\otimes M_{-1}\otimes M_{1}\otimes M_{\frac{3}{2}}\otimes M_{2} & \xrightarrow{1)\ \widehat{\alpha}^2} & M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{24}\mathbf{P}_{35}\mathbf{P}_{12}(\zeta^{-1}\mathbf{P}_{23}),\\
M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_{1} & \xrightarrow{2)\ \widehat{\beta}^2} & M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{15}\mathbf{P}_{13}\mathbf{A}_{1}^{2},\\
M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_{1} & \xrightarrow{3)\ \widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{13}\mathbf{P}_{34}\mathbf{P}_{23}(\zeta^{-1}\mathbf{P}_{35}),\\
M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & \xrightarrow{4)\ \widehat{\beta}} & M_{-1}\otimes M_{-\frac{3}{4}}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_{2},\\
M_{-1}\otimes M_{-\frac{3}{4}}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_{1} & \xrightarrow{5)\ \widehat{\alpha}} & M_{-4}\otimes M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{1} & :\ \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}),\\
M_{-4}\otimes M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{1} & \xrightarrow{6)\ \widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_{4},\\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & \xrightarrow{7)\ \widehat{\alpha}^2} & M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{1}\otimes M_{2} & :\ \mathbf{P}_{25}\mathbf{P}_{13}\mathbf{P}_{45}(\zeta^{-1}\mathbf{P}_{15}),\\
M_{-3}\otimes M_{-2}\otimes M_{-1}\otimes M_{1}\otimes M_{2} & \xrightarrow{8)\ \widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & :\ \mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}\mathbf{P}_{13}\mathbf{A}_{3},\\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_{\frac{1}{2}}\otimes M_{1} & \xrightarrow{9)\ \widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_{1}\otimes M_{2}\otimes M_{3} & :\ \mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{35}(\zeta^{-1}\mathbf{P}_{15}),\\
M_{-2}\otimes M_{-1}\otimes M_{1}\otimes M_{2}\otimes M_{3} & \xrightarrow{10)\ \widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_{-\frac{1}{4}}\otimes M_{1} & :\ \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_{2},\\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_{-\frac{1}{4}}\otimes M_{1} & \xrightarrow{11)\ \widehat{\alpha}} & M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-\frac{4}{3}}\otimes M_{-1}\otimes M_{1} & :\ \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}),\\
M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-\frac{4}{3}}\otimes M_{-1}\otimes M_{1} & \xrightarrow{12)\ \widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{1}\otimes M_{2}\otimes M_{3} & :\ \mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_{4}.
\end{array}
$$

and therefore by assembling all these we get the LHS of (4.40):

$$
\begin{aligned}
&(\mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\underbrace{\underline{\mathbf{P}_{14}}\mathbf{A}_{4})(\mathbf{P}_{12}\mathbf{P}_{13}\underline{\mathbf{P}_{14}}}(\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}))(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\underline{\mathbf{P}_{12}\mathbf{A}_{2})(\mathbf{P}_{14}\mathbf{P}_{12}}\mathbf{P}_{35}(\zeta^{-1}\mathbf{P}_{15}))\\
&\cdot(\mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}\underline{\mathbf{P}_{13}\mathbf{A}_{3})(\mathbf{P}_{25}\mathbf{P}_{13}}\ \underline{\mathbf{P}_{45}(\zeta^{-1}\mathbf{P}_{15}))(\mathbf{P}_{24}\mathbf{P}_{45}}\mathbf{P}_{34}\underline{\mathbf{P}_{14}\mathbf{A}_{4})(\mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}}(\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}))\\
&\cdot(\underline{\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_{2})(\mathbf{P}_{13}\mathbf{P}_{34}\mathbf{P}_{23}}(\zeta^{-1}\mathbf{P}_{35}))(\mathbf{P}_{14}\underline{\mathbf{P}_{12}\mathbf{P}_{15}\mathbf{P}_{13}\mathbf{A}_{1}^{2})(\mathbf{P}_{24}\mathbf{P}_{35}\mathbf{P}_{12}}(\zeta^{-1}\mathbf{P}_{23}))\\
&\overset{(4.22)}{=}\zeta^{-4}\underline{\mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}(\mathbf{A}_{1}\mathbf{P}_{42}}\mathbf{P}_{43})\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{23}\underline{\mathbf{P}_{24}\mathbf{P}_{25}(\mathbf{A}_{1}\mathbf{P}_{24})}\underline{\mathbf{P}_{35}\mathbf{P}_{15}\mathbf{P}_{35}}\mathbf{P}_{23}\mathbf{P}_{34}(\mathbf{A}_{1}\mathbf{P}_{25})\\
&\cdot(\mathbf{P}_{14}\mathbf{P}_{25})\underline{\mathbf{P}_{34}(\mathbf{A}_{1}\mathbf{P}_{42}\mathbf{P}_{43}})\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}(\mathbf{P}_{34}\mathbf{P}_{35}\mathbf{P}_{13}\mathbf{A}_{3}\mathbf{P}_{12}\mathbf{P}_{24})\underline{\mathbf{P}_{35}\underline{\cancel{\mathbf{P}_{14}}(\mathbf{P}_{25}\mathbf{P}_{23}\mathbf{A}_{2}^{2}\cancel{\mathbf{P}_{14}}}\mathbf{P}_{35})}\mathbf{P}_{23}\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{25}\underline{\mathbf{P}_{32}\mathbf{A}_{1})\mathbf{P}_{43}\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{23}}(\mathbf{P}_{45}\mathbf{A}_{1})(\mathbf{P}_{13})\mathbf{P}_{23}\mathbf{P}_{34}\mathbf{A}_{1}\underline{\cancel{\mathbf{P}_{25}}\mathbf{P}_{14}\cancel{\mathbf{P}_{25}}}\\
&\cdot(\mathbf{A}_{1}\mathbf{P}_{32})\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{34}\mathbf{P}_{35}\mathbf{P}_{13}\mathbf{A}_{3}\mathbf{P}_{12}\mathbf{P}_{24}(\underline{\mathbf{P}_{23}\mathbf{P}_{25}\mathbf{A}_{2}^{2})\mathbf{P}_{23}}\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{25}(\underline{\mathbf{A}_{1}\mathbf{P}_{42}}\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5})\mathbf{P}_{45}\mathbf{A}_{1}\mathbf{P}_{13}\mathbf{P}_{23}\underline{\mathbf{P}_{34}\mathbf{A}_{1}(\mathbf{P}_{14})\mathbf{A}_{1}\mathbf{P}_{32}\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{34}}\\
&\quad\cdot\underline{\mathbf{P}_{35}\mathbf{P}_{13}\mathbf{A}_{3}\mathbf{P}_{12}\mathbf{P}_{24}(\mathbf{P}_{35}}\mathbf{A}_{3}^{2})\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{25}\underline{\mathbf{P}_{42}\mathbf{A}_{1}^{2}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{45}}\mathbf{A}_{1}\underline{\mathbf{P}_{13}\mathbf{P}_{23}(\mathbf{A}_{1}\mathbf{P}_{13}}\mathbf{A}_{1}\underline{\mathbf{P}_{42}\mathbf{A}_{1}\mathbf{T}_{15}^{-1}\mathbf{A}_{5})(\mathbf{P}_{15}\mathbf{A}_{5}\mathbf{P}_{12}\mathbf{P}_{24}})\mathbf{A}_{3}^{2}\\
&\overset{(4.22)}{=}\zeta^{-4}\underline{\mathbf{P}_{25}(\mathbf{P}_{45}\mathbf{P}_{52}}\ \underline{\mathbf{A}_{1}^{2}\mathbf{T}_{14}^{-1}\mathbf{A}_{4})\mathbf{A}_{1}(\mathbf{P}_{21}}\mathbf{A}_{3})\underline{\mathbf{A}_{1}(\mathbf{A}_{1}}\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{15}\mathbf{A}_{5}\mathbf{P}_{14})\mathbf{A}_{3}^{2}\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{42})(\underline{\mathbf{P}_{21}\mathbf{A}_{2}^{2}\mathbf{T}_{24}^{-1}\mathbf{A}_{4}\mathbf{A}_{2})\mathbf{A}_{3}(\mathbf{A}_{1}^{2})\mathbf{T}_{15}^{-1}\mathbf{A}_{5}\mathbf{P}_{15}\mathbf{A}_{5}\mathbf{P}_{14}}\mathbf{A}_{3}^{2}\\
&\overset{(4.22)}{=}\zeta^{-4}\underline{\mathbf{P}_{42}(\mathbf{P}_{14}\mathbf{P}_{24}}\ \underline{\mathbf{A}_{2}^{2}\mathbf{T}_{21}^{-1}\mathbf{A}_{1}\mathbf{A}_{2}\mathbf{A}_{3}\mathbf{A}_{4}^{2}\mathbf{T}_{45}^{-1}\mathbf{A}_{5}\mathbf{P}_{45}}\mathbf{A}_{5})\mathbf{A}_{3}^{2}\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{12})(\mathbf{P}_{45}\mathbf{A}_{2}^{2}\mathbf{T}_{21}^{-1}\mathbf{A}_{1}\mathbf{A}_{2}\ \underbrace{\underline{\mathbf{A}_{3}}\ \underline{\mathbf{A}_{5}^{2}\mathbf{T}_{54}^{-1}\mathbf{A}_{4})\mathbf{A}_{5}}}\ \mathbf{A}_{3}^{2}\\
&=\zeta^{-4}\mathbf{P}_{12}\mathbf{P}_{45}(\mathbf{A}_{2}^{2}\mathbf{T}_{21}^{-1}\mathbf{A}_{1}\mathbf{A}_{2})(\mathbf{A}_{5}^{2}\mathbf{T}_{54}^{-1}\mathbf{A}_{4}\mathbf{A}_{5})\mathbf{A}_{3}^{3}.
\end{aligned}
$$

We do likewise for the RHS of (4.40); we break up the RHS of (4.40) from the right into twelve steps: 1) $\widehat{\beta}$, 2) $\widehat{\alpha}$, 3) $\widehat{\beta}$, 4) $\widehat{\alpha}^2$, 5) $\widehat{\beta}^2$, 6) $\widehat{\alpha}^2$, 7) $\widehat{\beta}$, 8) $\widehat{\alpha}$, 9) $\widehat{\beta}$, 10) $\widehat{\alpha}^2$, 11) $\widehat{\beta}$, 12) $\widehat{\alpha}^2$. Ignore the hats and think of $\widehat{\alpha}, \widehat{\beta}$ as α, β, and we keep track of the action of each step on marked tessellations, starting from the standard one τ^*_{mark}; this process is depicted in Fig. 15.

As in the LHS of (4.40), the RHS operator $(\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2)(\widehat{\beta}\widehat{\alpha}\widehat{\beta})$ acts in an interesting way only on the factors $M_{-2}, M_{-1}, M_1, M_{\frac{3}{2}}, M_2$. We complete the list of operators for each stage of the RHS of (4.40) (at each stage, tensor product is ordered so that $\mathbb{Q}^\times$-subscripts are in the increasing order):

$$
\begin{array}{rcll}
M_{-2}\otimes M_{-1}\otimes M_1\otimes M_{\frac{3}{2}}\otimes M_2 & \xrightarrow{1)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{2}{5}}\otimes M_{-\frac{1}{3}}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_2, \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{2}{5}}\otimes M_{-\frac{1}{3}}\otimes M_1 & \xrightarrow{2)\,\widehat{\alpha}} & M_{-2}\otimes M_{-\frac{5}{3}}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_1\mathbf{T}_{15}^{-1}\mathbf{A}_5), \\
M_{-2}\otimes M_{-\frac{5}{3}}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1 & \xrightarrow{3)\,\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_{\frac{3}{2}}\otimes M_2 & : \mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4, \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_{\frac{3}{2}}\otimes M_2 & \xrightarrow{4)\widehat{\alpha}^2} & M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_2 & : \mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{13}), \\
M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_2 & \xrightarrow{5)\widehat{\beta}^2} & M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1 & : \mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{15}\mathbf{P}_{13}\mathbf{A}_1^2, \\
M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_1 & \xrightarrow{6)\widehat{\alpha}^2} & M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_{\frac{2}{3}}\otimes M_1 & : \mathbf{P}_{13}\mathbf{P}_{34}\mathbf{P}_{23}(\zeta^{-1}\mathbf{P}_{35}), \\
M_{-2}\otimes M_{-1}\otimes M_{\frac{1}{2}}\otimes M_{\frac{2}{3}}\otimes M_1 & \xrightarrow{7)\widehat{\beta}} & M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{3}{5}}\otimes M_{-\frac{1}{2}}\otimes M_1 & : \mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_2, \\
M_{-1}\otimes M_{-\frac{2}{3}}\otimes M_{-\frac{3}{5}}\otimes M_{-\frac{1}{2}}\otimes M_1 & \xrightarrow{8)\widehat{\alpha}} & M_{-3}\otimes M_{-\frac{5}{2}}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & : \mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_1\mathbf{T}_{15}^{-1}\mathbf{A}_5), \\
M_{-3}\otimes M_{-\frac{5}{2}}\otimes M_{-2}\otimes M_{-1}\otimes M_1 & \xrightarrow{9)\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_{\frac{2}{3}}\otimes M_1 & : \mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4, \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{\frac{1}{2}}\otimes M_{\frac{2}{3}}\otimes M_1 & \xrightarrow{10)\widehat{\alpha}^2} & M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1\otimes M_2 & : \mathbf{P}_{25}\mathbf{P}_{13}\mathbf{P}_{45}(\zeta^{-1}\mathbf{P}_{15}), \\
M_{-2}\otimes M_{-\frac{3}{2}}\otimes M_{-1}\otimes M_1\otimes M_2 & \xrightarrow{11)\widehat{\beta}} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_1\otimes M_2 & : \mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}\mathbf{P}_{13}\mathbf{A}_3, \\
M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_{-\frac{1}{3}}\otimes M_1\otimes M_2 & \xrightarrow{12)\widehat{\alpha}^2} & M_{-1}\otimes M_{-\frac{1}{2}}\otimes M_1\otimes M_2\otimes M_3 & : \mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14}).
\end{array}
$$

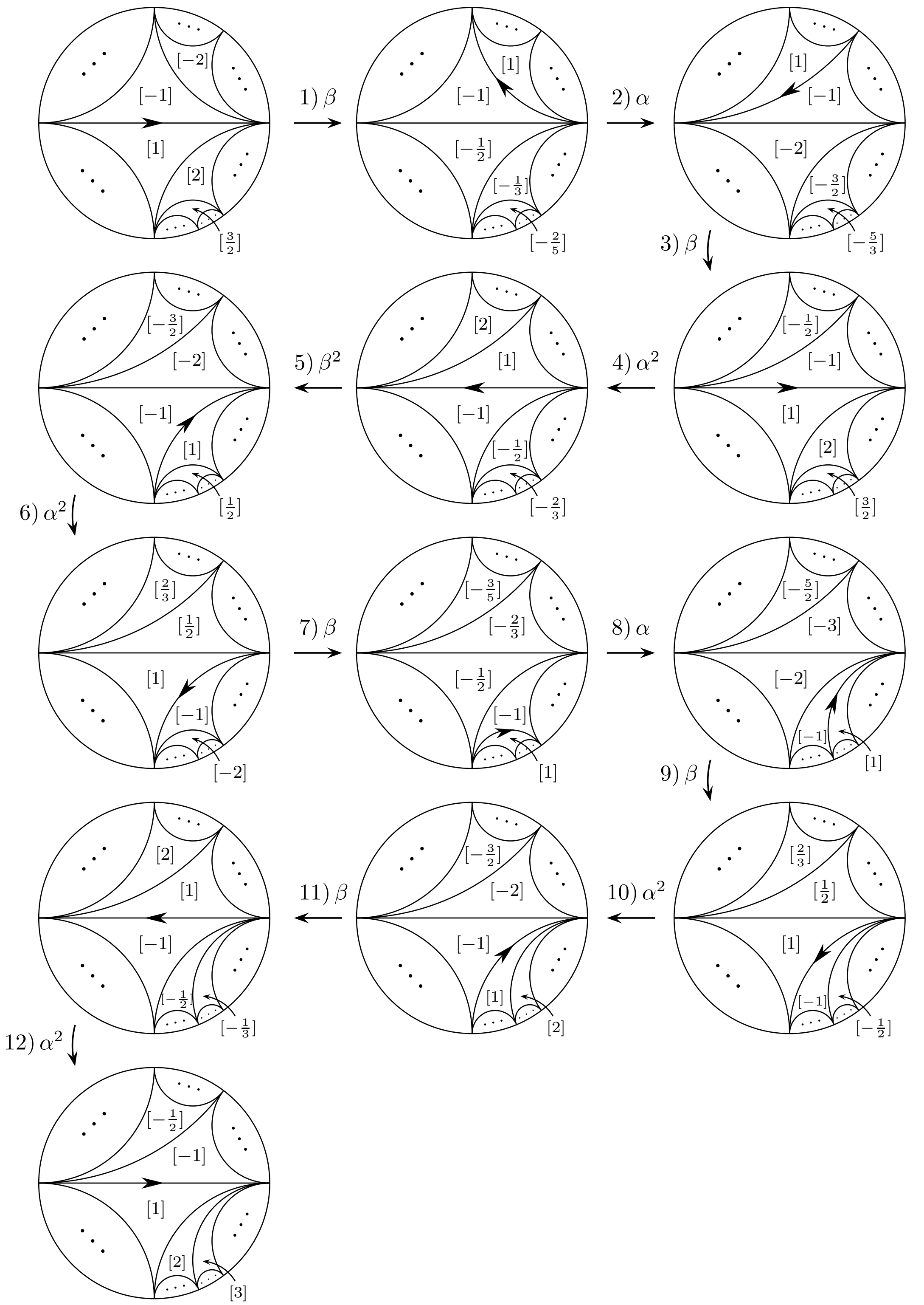


FIGURE 15. The RHS of (4.40), the action of $\alpha^2\beta\alpha^2\beta\alpha\beta\alpha^2\beta^2\alpha^2\,\beta\alpha\beta$ on τ^*_{mark}

Putting these together, we get the RHS of (4.40):

$$
\begin{aligned}
&(\mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{14}))(\mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}\ \mathbf{P}_{13}\mathbf{A}_3)(\mathbf{P}_{25}\mathbf{P}_{13}\ \mathbf{P}_{45}(\zeta^{-1}\mathbf{P}_{15}))(\mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4)\\
&\cdot(\mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_1\mathbf{T}_{15}^{-1}\mathbf{A}_5))(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_2)(\mathbf{P}_{13}\mathbf{P}_{34}\mathbf{P}_{23}(\zeta^{-1}\mathbf{P}_{35}))(\mathbf{P}_{14}\mathbf{P}_{12}\mathbf{P}_{15}\mathbf{P}_{13}\mathbf{A}_1^2)\\
&\cdot(\mathbf{P}_{35}\mathbf{P}_{23}\mathbf{P}_{34}(\zeta^{-1}\mathbf{P}_{13}))(\mathbf{P}_{24}\mathbf{P}_{45}\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4)(\mathbf{P}_{12}\mathbf{P}_{13}\mathbf{P}_{14}(\mathbf{A}_1\mathbf{T}_{15}^{-1}\mathbf{A}_5))(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{P}_{25}\mathbf{P}_{12}\mathbf{A}_2)\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{24}\cancel{\mathbf{P}_{45}}(\mathbf{P}_{13}\cancel{\mathbf{P}_{45}}\mathbf{P}_{24})(\mathbf{A}_1\cancel{\mathbf{P}_{25}})(\mathbf{P}_{14}\cancel{\mathbf{P}_{25}})\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4\\
&\qquad\cdot(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{13}\mathbf{P}_{14}\mathbf{P}_{15})\mathbf{A}_2(\mathbf{P}_{14}\mathbf{P}_{21}\mathbf{P}_{15}\mathbf{P}_{34}\mathbf{P}_{32}\mathbf{P}_{35})\mathbf{A}_1^2\\
&\qquad\cdot\mathbf{P}_{35}\cancel{\mathbf{P}_{23}}(\mathbf{P}_{14}\cancel{\mathbf{P}_{23}}\mathbf{P}_{35})\mathbf{P}_{14}\mathbf{A}_4(\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{13}\mathbf{P}_{14}\mathbf{P}_{15})\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{13})\mathbf{A}_1(\mathbf{P}_{14})\mathbf{P}_{34}\mathbf{P}_{14}\mathbf{A}_4\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{13}\\
&\qquad\cdot(\mathbf{P}_{45}\mathbf{A}_2)\mathbf{P}_{21}\mathbf{P}_{15}\mathbf{P}_{34}\mathbf{P}_{32}\mathbf{P}_{35}\mathbf{A}_1^2(\mathbf{P}_{14})(\mathbf{A}_1\mathbf{P}_{23}\mathbf{P}_{21}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{43})\mathbf{P}_{15}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{13}\mathbf{A}_1(\mathbf{P}_{31})\mathbf{A}_4\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{13}\\
&\qquad\cdot\mathbf{P}_{45}\mathbf{A}_2\mathbf{P}_{21}\mathbf{P}_{15}\mathbf{P}_{34}(\mathbf{P}_{25}\mathbf{A}_1^2\mathbf{P}_{14}\mathbf{A}_1)\mathbf{P}_{21}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{43}\mathbf{P}_{15}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{A}_3)\mathbf{A}_4\mathbf{P}_{23}\mathbf{P}_{24}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5\mathbf{P}_{13}\mathbf{P}_{45}\mathbf{A}_2\mathbf{P}_{21}\mathbf{P}_{15}(\mathbf{P}_{25}\mathbf{A}_1^2\mathbf{P}_{13}\mathbf{A}_1\mathbf{P}_{21}\mathbf{A}_2\mathbf{T}_{25}^{-1}\mathbf{A}_5)\mathbf{P}_{15}\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{45}\mathbf{A}_3\mathbf{A}_5\mathbf{P}_{23}\mathbf{P}_{25}\mathbf{A}_2\mathbf{T}_{24}^{-1}\mathbf{A}_4\mathbf{P}_{13})\mathbf{A}_2\cancel{\mathbf{P}_{21}}(\cancel{\mathbf{P}_{21}}\mathbf{A}_5^2\mathbf{P}_{53}\mathbf{A}_5\mathbf{P}_{25}\mathbf{A}_2\mathbf{T}_{21}^{-1}\mathbf{A}_1)\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{45}\mathbf{A}_3\mathbf{A}_5\mathbf{P}_{23}(\mathbf{A}_5\mathbf{T}_{54}^{-1}\mathbf{A}_4\mathbf{P}_{13}\mathbf{A}_5\mathbf{A}_2^2\mathbf{P}_{23}\ \mathbf{A}_2)\mathbf{A}_2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{45}\mathbf{A}_3\mathbf{A}_5(\mathbf{A}_5\mathbf{T}_{54}^{-1}\mathbf{A}_4\mathbf{P}_{12}\mathbf{A}_5\mathbf{A}_3^2)(\mathbf{A}_2^2)\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{45}\mathbf{A}_3\mathbf{A}_5(\mathbf{A}_5\mathbf{T}_{54}^{-1}\mathbf{A}_4\mathbf{P}_{12}\mathbf{A}_5\ \underbrace{\mathbf{A}_3^2)(\mathbf{A}_2^2)\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2}\\
&\overset{(4.22)}{=}\zeta^{-4}(\mathbf{P}_{12}\mathbf{P}_{45}\ \underbrace{\mathbf{A}_3\ (\mathbf{A}_5^2)\mathbf{T}_{54}^{-1}\mathbf{A}_4)\mathbf{A}_5(\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2}\ \mathbf{A}_3^2)\\
&\overset{(4.22)}{=}\zeta^{-4}\mathbf{P}_{12}\mathbf{P}_{45}(\underbrace{\mathbf{A}_5^2\mathbf{T}_{54}^{-1}\mathbf{A}_4\mathbf{A}_5\ \mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2}\ \mathbf{A}_3)\mathbf{A}_3^2\\
&=\zeta^{-4}\mathbf{P}_{12}\mathbf{P}_{45}(\mathbf{A}_2^2\mathbf{T}_{21}^{-1}\mathbf{A}_1\mathbf{A}_2)(\mathbf{A}_5^2\mathbf{T}_{54}^{-1}\mathbf{A}_4\mathbf{A}_5)(\mathbf{A}_3^3),
\end{aligned}
$$

which by inspection is identical to the resulting expression for the LHS of (4.40). □

Combining the Propositions 4.12, 4.13, 4.14 and 4.15, we get that the images $\widehat{\alpha}$ and $\widehat{\beta}$ of α and β under the almost G_{mark}-homomorphism $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$ (see Cor. 3.12 and (4.17)), as unitary operators on $\mathscr{M}$, satisfy

$$
\begin{aligned}
&(\widehat{\beta}\widehat{\alpha})^5 = \zeta^{-3}, \qquad \widehat{\alpha}^4 = \zeta^{-2}, \qquad \widehat{\beta}^3 = 1,\\
&[\ \widehat{\beta}\widehat{\alpha}\widehat{\beta},\ \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\ ] = [\ \widehat{\beta}\widehat{\alpha}\widehat{\beta},\ \widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}\widehat{\alpha}\widehat{\beta}\widehat{\alpha}^2\widehat{\beta}^2\widehat{\alpha}^2\ ] = 1,
\end{aligned} \tag{4.42}
$$

where ζ is a complex number of modulus 1 defined in (3.34) (depending on the quantization parameter $b \in \mathbb{R}$). Thus it is easy to see that $\rho^{Kash}(R_{mark})$ is contained in $\mathbb{C}^*$ and is generated by ζ, where R_{mark} is the normal subgroup of the free group F_{mark} generated by the relations of α and β for the group G_{mark} (see (3.5)).

Following the method of the constructing the central extension from an almost group homomorphism which is described in §4.1, the central extension $\widehat{G}_{mark}^{Kash}$ of G_{mark} thus obtained from

the almost G_{mark}-homomorphism ρ^{Kash} has the presentation with generators $\bar{\alpha}, \bar{\beta}, z$ with the following relations, if we let the central generator z correspond to ζ^{-1}:

$$\begin{array}{l} (\bar{\beta}\bar{\alpha})^5 = z^3, \qquad \bar{\alpha}^4 = z^2, \qquad \bar{\beta}^3 = 1, \\ \big[\bar{\beta}\bar{\alpha}\bar{\beta},\, \bar{\alpha}^2\bar{\beta}\bar{\alpha}\bar{\beta}\bar{\alpha}^2\big] = \big[\bar{\beta}\bar{\alpha}\bar{\beta},\, \bar{\alpha}^2\bar{\beta}\bar{\alpha}^2\bar{\beta}\bar{\alpha}\bar{\beta}\bar{\alpha}^2\bar{\beta}^2\bar{\alpha}^2\big] = [\bar{\alpha}, z] = \big[\bar{\beta}, z\big] = 1, \end{array} \tag{4.43}$$

hence is isomorphic to the group $T_{3,2,0,0}$ appearing in Thm. 4.7, thus proving (4.13). From Thm. 4.7, it's easy to compute the extension class of this extension, which turns out to be $6\chi \in H^2(T)$. This proves Thm. 4.9.

This proof is quite algebraic, and we had to check all the α, β-relations of G_{mark} using the operators $\widehat{\alpha}, \widehat{\beta}$ (Def. 4.10), and see how the relations are lifted. There wasn't much of topology; the Figures 12, 13, 14 and 15 are mainly to keep track of the triangle label permutations. This algebraic proof may be a useful exercise for the application to different surfaces (like finite type surfaces), but it seems to obscure the topological nature of the whole story about the Ptolemy-Thompson group $G_{mark} \cong T$.

Meanwhile, Funar and Sergiescu [FuS] identified the central extension $\widehat{G}^{CF}_{mark}$ (which they denoted by $\widehat{T}$, the 'dilogarithmic central extension'), induced by Chekhov-Fock's almost linear representation ρ^{CF} of G_{mark}, with the group T^*_{ab}, the 'relative abelianization' of (one version of) the braided Ptolemy-Thompon group T^* (for T^*_{ab}, see [FuS] and Funar-Kapoudjian [FuKap2] for details); see (5.11). The group T^* naturally arises as the mapping class group of some topological structure of a 'ribbon graph', or equivalently, of a (Farey-type) tessellation. It is remarkable that the almost linear representation ρ^{CF} of G_{mark} precisely captures this topological information (or vice versa).

Now, we can ask the following question: can the central extension $\widehat{G}^{Kash}_{mark}$ induced by Kashaev's almost linear representation ρ^{Kash} be identified with some central extension of $T \cong G_{mark}$ which has a topological nature? The answer is yes, and $\widehat{G}^{Kash}_{mark}$ can be identified with the group $T^\sharp_{ab}$, whose description is quite close to that of T^*_{ab} (see [FuKap2] for $T^\sharp_{ab}$ and $T^\sharp$); see (5.12).

We shall give a proof of the direct identification of $\widehat{G}^{Kash}_{mark}$ with $T^\sharp_{ab}$, which together with Funar-Kapoudjian's result $T^\sharp_{ab} \cong T_{3,2,0,0}$ (mentioned in [FuS], which is a direct consequence of [FuKap2]) yields Thm. 4.9. Thus we call this investigation the 'topological proof of Theorem 4.9', which is the subject of the following subsection.

It was Funar [Fu] who informed the author of the result $T^\sharp_{ab} \cong T_{3,2,0,0}$ and suggested to seek for a 'geometric' identification of $\widehat{G}^{Kash}_{mark}$ and $T^\sharp_{ab}$.

5. Topological proof of Theorem 4.9

In this section, we first review Funar and Kapoudjian's formulation ([FuKap2]) of asymptotically rigid mapping class groups T^* and $T^\sharp$ of the punctured ribbon trees, and their relative abelianizations T^*_{ab} and $T^\sharp_{ab}$. Then we introduce the punctured versions of the groups G_{mark} and G_{dot} (and also their relative abelianizations) and study the relationship between them, and also build a dual relationship between the ribbon tree model and the unit disc tessellation model. We will then finally give a 'graphical' proof of the identification of $\widehat{G}^{Kash}_{mark}$ with $T^\sharp_{ab}$.

5.1. **The extensions T^*, $T^\sharp$ of the Ptolemy-Thompson group T by the braid group B_∞.** The group that is dubbed 'Thomson group' has different guises: the group of dyadic piecewise affine homeomorphisms of $[0,1]$, the group of dyadic piecewise affine homeomorphisms of the circle $[0,1]/0 \sim 1$, the group of flips, the group $PPSL(2,\mathbb{Z})$ of piecewise-$PSL(2,\mathbb{Z})$

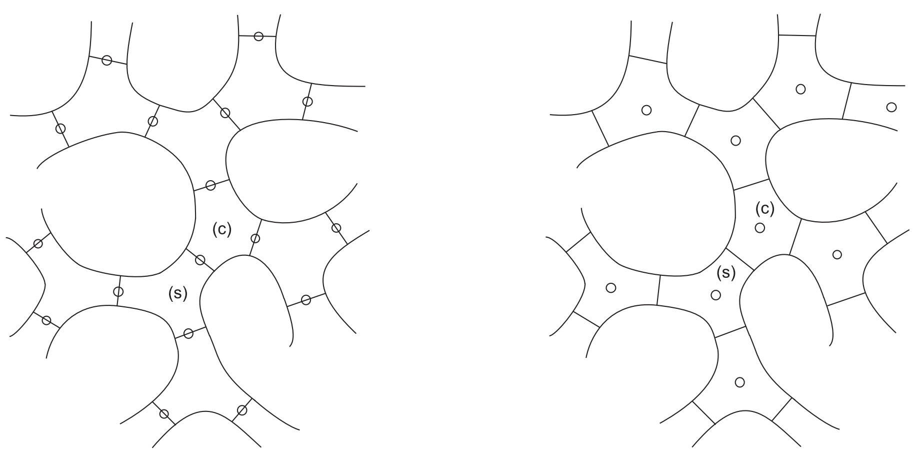
homeomorphisms of the circle (see Def. 2.13), and as a mapping class group (see [FuS], for example). Here we call it *the Ptolemy-Thompson group* T, given by generators and relations exactly same as G_{mark} in (2.9). In this section we use its topological definition, namely as the mapping class group of some infinite surface (called the 'ribbon tree'), as done in the works of Funar, Kapoudjian, Sergiescu, Kashaev, and collaborators; see [FuKapS] for a survey. We will borrow the definitions from [FuS].

The surfaces appearing here will be oriented and all the homeomorphisms considered will be orientation-preserving, and all the actions are the left actions.

Definition 5.1 (See e.g. [FuKap2])**.** *The* ribbon tree D *is the planar surface obtained by thickening the infinite binary tree in the plane. We denote by* D^* *(respectively,* $D^\sharp$*) the ribbon tree with infinitely many punctures (depicted as* $\circ$ *in the pictures), one puncture for each edge (resp. each vertex) of the tree. A homeomorphism of* D^* *(resp.* $D^\sharp$*) is a homeomorphism of* D *which permutes the punctures* D^* *(resp.* $D^\sharp$*).*

Remark 5.2. *Here the infinite binary tree means the same thing as the infinite trivalent graph without cycles. But calling it the infinite binary tree is more suggestive in some sense; see Rem. 5.6.*

Definition 5.3. *A* rigid structure *on* D*,* D^* *or* $D^\sharp$ *is a decomposition into hexagons by means of a family of arcs whose endpoints are on the boundary of* D*. In the case of* D^**, each hexagon contains no puncture within its interior but each arc passes through a unique puncture. In the case of* $D^\sharp$*, each hexagon contains exactly one puncture in its interior. It is assumed that these arcs are pairwise non-homotopic in* D*, by the homotopies keeping the endpoints of the arcs on the boundary of* D*. The choice of a rigid structure of reference is called the* canonical rigid structure*. The canonical rigid structure of the ribbon tree* D *is such that each arc of this rigid structure crosses once and transversely a unique edge of the tree; see Fig. 16, ignoring the punctures and labels. The canonical rigid structures on* D^* *and* $D^\sharp$ *are assumed to coincide with the canonical rigid structure of* D *when forgetting the punctures; see Fig. 16.*



(A) The canonical marked rigid structure on D^* (B) The canonical marked rigid structure on $D^\sharp$

FIGURE 16. The canonical marked rigid structures

Definition 5.4. *1. Let $D^\diamondsuit$ denote D, D^* or $D^\sharp$. A planar subsurface of $D^\diamondsuit$ is* admissible *if it is a connected finite union of hexagons belonging to the canonical rigid structure (we mean that this finite union is same as some finite union of hexagons from the canonical rigid structure) The* frontier *of an admissible surface is the union of the arcs contained in the boundary. The remaining arcs will be called the* separating arcs.
2. Let φ be a homeomorphism of $D^\diamondsuit$. One says that φ is asymptotically rigid *if the following conditions are fulfilled:*

(1) *There exists an admissible subsurface $\Sigma \subset D^\diamondsuit$ such that $\varphi(\Sigma)$ is also admissible.*
(2) *The complement $D^\diamondsuit - \Sigma$ is a union of n infinite surfaces. Then the restriction $\varphi : D^\diamondsuit - \Sigma \to D^\diamondsuit - \varphi(\Sigma)$ is* rigid*, meaning that it respects the canonical rigid structures in the complements of the compact subsurfaces, mapping hexagons into hexagons. Such a surface Σ is called a* support *for φ.*

One denotes by T, T^* and $T^\sharp$ the groups of isotopy classes of asymptotically rigid homeomorphisms of D, D^* and $D^\sharp$, respectively.

It is legitimate to denote by T the (asymptotically rigid) mapping class group of D, because there is an isomorphism from the finitely presented group $T \cong G_{mark}$ in (2.9) to this mapping class group (widely developed in [FuKap1] and [KapS]). We find it necessary to use the following decoration of the rigid structures when describing the generators of this mapping class group T, although it's not usually mentioned:

Definition 5.5. *Let $D^\diamondsuit$ denote D, D^* or $D^\sharp$. A* marked rigid structure *on $D^\diamondsuit$ is a rigid structure on $D^\diamondsuit$ together with the choice of a pair of two adjacent hexagons, each labeled by (c) and (s) (where (c) stands for 'central', (s) for 'secondary'). See Fig. 16.*

Remark 5.6. *If we only have a rigid structure on $D^\diamondsuit$ without a marking (i.e. choice of two adjacent hexagons (c) and (s)), then collapsing the ribbon (fat) graph to thin graph yields an infinite trivalent tree, that is, there is no distinguished vertex or edge. On the other hand, if we collapse a ribbon graph with a marked rigid structure, then we obtain an infinite trivalent graph with two distinguished adjacent vertices. If we consider adding a vertex in the middle of the edge connecting these two vertices, then we get a infinite binary tree with this added vertex being the 'root'.*

We now assume that any choice of the rigid structure comes equipped with a marking, i.e. the choice of two adjacent hexagons labeled by (c) and (s). The isomorphism from the finitely presented group $T \cong G_{mark}$ (2.9) to the mapping class group of asymptotically rigid homeomorphisms of D is described by the images of α, β as follows:

- The α homeomorphism (class) maps the union of the hexagons (c) and (s) (which are always adjacent to each other, whose union can be viewed as an octagon) to itself by rotating it 'by the angle $\frac{\pi}{2}$', permuting the four branches of the ribbon tree issued from this union. See Fig. 17 (the labels $(1),(2),(3),(4)$ are just for illustration).

  Note that α is not globally rigid, but α^2 is. ('globally rigid' means that it respects the rigid structure)

- The β homeomorphism (class) maps the hexagon (c) to itself by rotating it counterclockwise 'by the angle $\frac{\pi}{3}$'. Further, it acts as the corresponding counterclockwise rotation of order three which permutes the three branches of the ribbon tree issued from the hexagon (c). See Fig. 18 (the labels $(1),(2),(3),(4)$ are just for illustration).

  In fact, this mapping class is globally rigid.

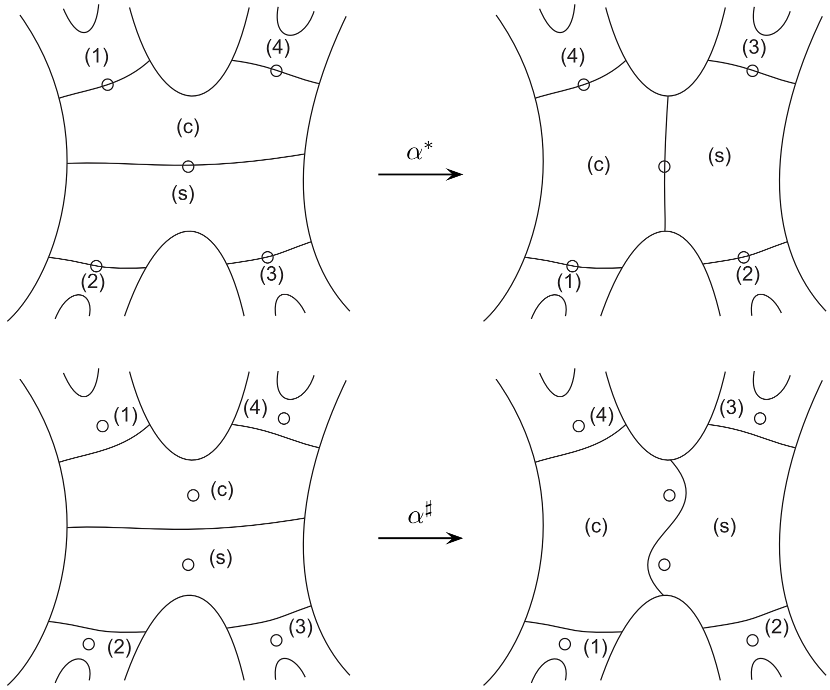


FIGURE 17. The actions of $\alpha^*, \alpha^\sharp$ on $D^*, D^\sharp$ (forget the punctures for the α action on D)

Proposition 5.7 (Lochak-Schneps [LoSc], typo corrected in [FuKap2]). *The above map from the finitely presented group G_{mark} (as in* (2.9)*) to the (asymptotically rigid) mapping class group T of D is an isomorphism.*

Remark 5.8. *In fact, in the proof of Prop. 5.7, Lochak-Schneps used marked (Farey-type) tessellations in their proof, which is dual to the ribbon tree D. See §5.3 of the present paper for this dual correspondence.*

Hence, indeed the notation T for the mapping class group of D is legit; we just write α, β for the elements of this mapping class group, by abuse of notation. In a similar manner, we describe the mapping classes α^*, β^* of T^* (resp. $\alpha^\sharp, \beta^\sharp$ of $T^\sharp$) as follows:

- The mapping class α^* of T^* (resp. $\alpha^\sharp$ of $T^\sharp$) rotates the union of the two hexagons (c) and (s) counterclockwise 'by the angle $\frac{\pi}{2}$', keeping fixed the central puncture on the arc separating (c) and (s) (resp. fixing the two punctures in the interiors of (c) and (s), while rotating the arc separating (c) and (s) counterclockwise not allowing it to pass through those two punctures). It acts as the corresponding rotation on the four branches of the ribbon tree issued from this union; see Fig. 17. One has $(\alpha^*)^4 = 1$ while $(\alpha^\sharp)^4 = \sigma^2$, where σ denotes the 'braid' that permutes the punctures of (c) and (s); see Def. 5.9 for the braids.
- The mapping class β^* of T^* (resp. $\beta^\sharp$ of $T^\sharp$) leaves the hexagon (c) invariant by rotating it counterclockwise 'by the angle $\frac{\pi}{3}$'. Further β^* (resp. $\beta^\sharp$) acts as the corresponding

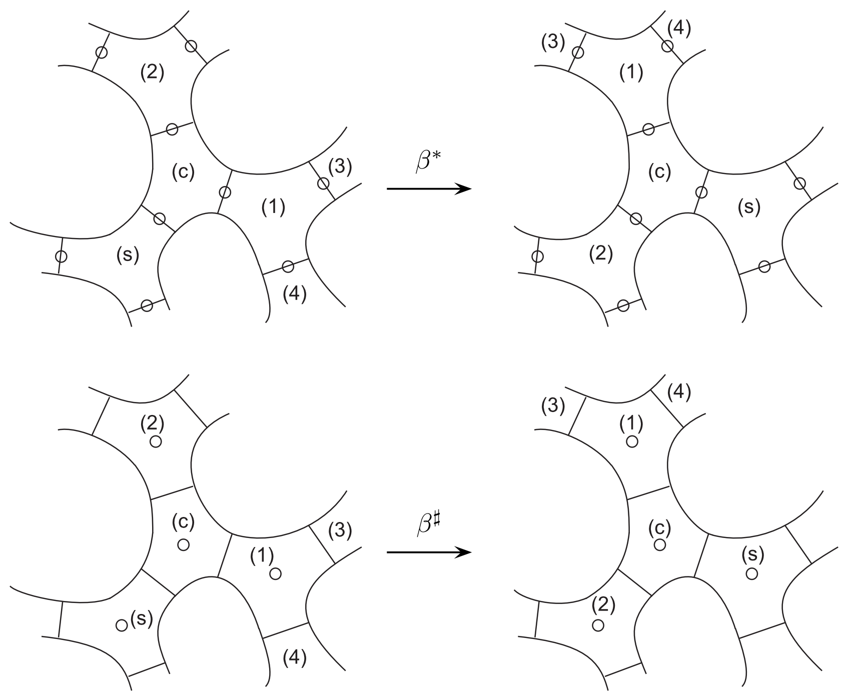


FIGURE 18. The actions of $\beta^*, \beta^\sharp$ on $D^*, D^\sharp$ (forget the punctures for the β action on D)

counterclockwise rotation of order three which permutes the three branches of the ribbon tree issued from the hexagon (c), cyclically permuting the punctures accordingly. See Fig. 18. One has $(\beta^*)^3 = 1$ (resp. $(\beta^\sharp)^3 = 1$).

Besides the above mapping classes, T^* and $T^\sharp$ have the following additional elements coming from the punctures, called the 'braidings'.

Definition 5.9. *Let e be an arc in D^* or $D^\sharp$ which connects two distinct punctures. We associate a* braiding *σ_e to e by considering the homeomorphism that moves clockwise the punctures at the endpoints of the edge e in a small neighborhood of the edge, in order to interchange their positions; see Fig. 19. This means that if γ is an arc transverse to e, then the braiding σ_e moves γ on the left when it approaches e. See Fig. 20; there, the homeomorphism (class) σ_e exchanges the punctures p and q, but still, the left region (hexagon) is (1) and right region is (2), because the homeomorphism σ_e acts as the identity outside a neighborhood of the edge e connecting p and q. Such a braiding will be called* positive, *while σ_e^{-1} is* negative.

In $D^\sharp$, for the edge e connecting the punctures of (c) and (s) which traverses exactly one separating arc (Def. 5.4), we let $\sigma = \sigma_e$.

For D^* and $D^\sharp$ respectively, we now define the group generated by the 'simple' braidings:

Definition 5.10. *An arc e in (the interior of) D^* (resp. in $D^\sharp$) connecting two distinct punctures is called* simple *if it does not intersect with any of the separating arcs (Def. 5.4)*

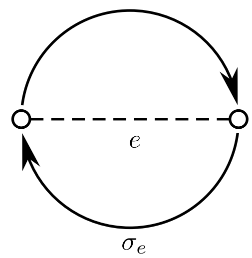


FIGURE 19. The braiding σ_e for the edge e

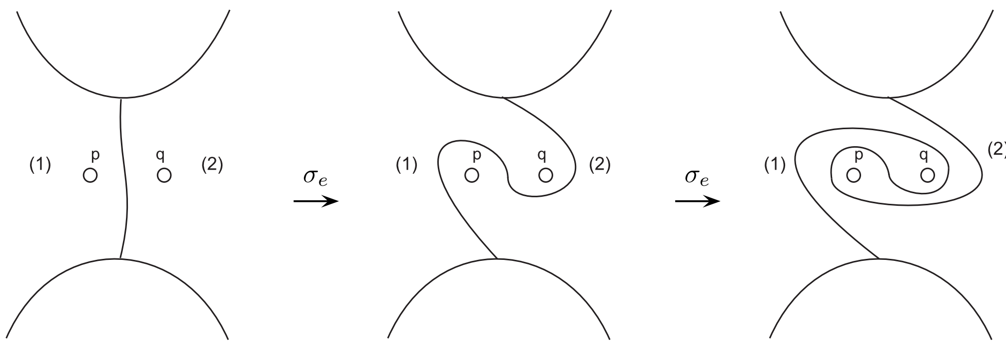


FIGURE 20. Examples of the action of braiding homeomorphism σ_e, where e is the edge connecting the punctures p and q

(resp. if it intersects with exactly one separating arc). For a simple arc e, we call the associated braiding σ_e a simple braiding. *Let $B(D^*)$ (resp. $B(D^\sharp)$) be the subgroup of T^* (resp. $T^\sharp$) (Def. 5.4) generated by all the simple braidings.*

From the definition, it's easy to see that a simple arc in D^* (resp. in $D^\sharp$) necessarily connects two punctures on the sides of a single hexagon (resp. the two punctures for the two adjacent hexagons). As in [FuKap2], each of the groups $B(D^*)$ and $B(D^\sharp)$ can also be viewed as the inductive limit of the group generated by the simple braidings in a finite subsurface containing only n punctures, as n tends to ∞ (we take a larger subsurface at each step). This smaller braid group for n punctures is actually isomorphic to the Artin's braid group B_n of n strands (pointed out to the author by Funar [Fu]). The braid groups associated to a graph (instead of just for the linearly aligned points as in the case of Artin's braid groups) was first considered by Sergiescu in [S], who proved that the braidings for the simple arcs (edges of a graph) generate the group of all possible braidings, and also found the relations holding among these simple braidings. We now have the following.

Definition 5.11. *Let the* infinite braid group B_∞ *be the inductive limit of Artin's braid groups B_n for n strands, as $n \to \infty$.*

Proposition 5.12. *The groups $B(D^*)$ and $B(D^\sharp)$ are isomorphic to B_∞, and these isomorphisms yield the embeddings $B_\infty \to T^*$ and $B_\infty \to T^\sharp$.*

Funar and Kapoudjian studied the presentations of T^* and $T^\sharp$ in the above described generators.

Theorem 5.13 ([FuKap2])**.** *The group T^* is generated by α^* and β^*, and*

$$(\beta^* \alpha^*)^5 = \sigma_{02} \tag{5.1}$$

holds, where σ_{02} is the braid for the simple arc (Def. 5.10) connecting the punctures 0 *and* 2 *in Fig. 21. Moreover, the group T^* can be finitely presented with the generators α^*, β^*.*

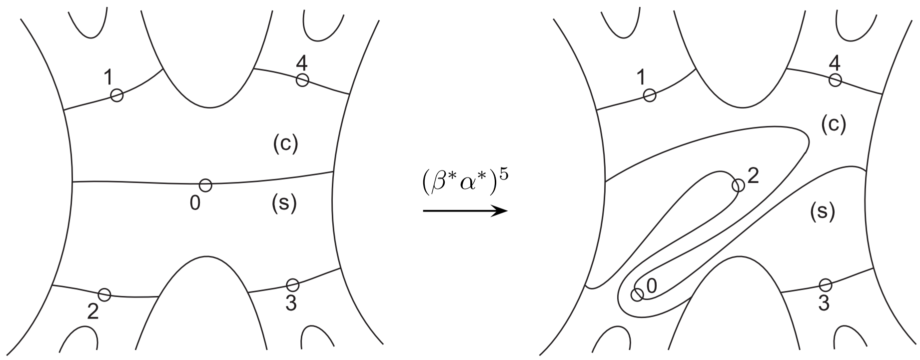


FIGURE 21. The action of $(\beta^*\alpha^*)^5$ on D^*

The group $T^\sharp$ is generated by $\alpha^\sharp$, $\beta^\sharp$ and the braiding σ (see Def. 5.9 for σ), and has a finite presentation with these three generators.

However, we'll only give a presentation of some quotient group of $T^\sharp$ in the next subsection (see [FuKap2] for the full presentation of the group $T^\sharp$).

By sending $\alpha^*, \beta^*, \sigma$ to $\alpha, \beta, 1$, we get the projection $T^* \to T$ and the following exact sequence

$$1 \longrightarrow B_\infty \longrightarrow T^* \longrightarrow T \longrightarrow 1 \tag{5.2}$$

Similarly, by sending $\alpha^\sharp, \beta^\sharp$ to α, β we get another exact sequence

$$1 \longrightarrow B_\infty \longrightarrow T^\sharp \longrightarrow T \longrightarrow 1. \tag{5.3}$$

So we can think of T^* and $T^\sharp$ as extensions of the Ptolemy-Thompson group T by the infinite braid group B_∞ (see Def. 5.10, Def. 5.11 and Prop. 5.12 for B_∞). Thus Funar and collaborators call both of $T^*, T^\sharp$ the *braided Ptolemy-Thompson groups*, although we won't be using this terminology in a serious manner. However, the two groups T^* and $T^\sharp$ are distinct, for example because they have different abelianizations:

$$H_1(T^*) \cong \mathbb{Z}/12\mathbb{Z}, \quad H_1(T^\sharp) \cong \mathbb{Z}/6\mathbb{Z}. \tag{5.4}$$

See Funar-Kapoudjian [FuKap2] for the detailed study of the relationship between T^* and $T^\sharp$.

Remark 5.14 (due to Frenkel [Fr])**.** *Note that $PSL(2,\mathbb{Z})$ is a small subgroup of T^* and $T^\sharp$. One may ask if* (5.4) *is related to the facts $H_1(SL(2,\mathbb{Z})) \cong \mathbb{Z}/12\mathbb{Z}$ and $H_1(PSL(2,\mathbb{Z})) \cong \mathbb{Z}/6\mathbb{Z}$.*

5.2. **The relative abelianizations $T^*_{ab}, T^\sharp_{ab}$ of the braided Ptolemy-Thompson groups $T^*, T^\sharp$.** We'd like to abelianize the kernel B_∞ (Def. 5.11) of the maps $T^* \to T$ in (5.2) and $T^\sharp T$ in (5.3). The relations holding between the simple braidings (Def. 5.10), i.e. the presentation of B_∞, are shown in [S]; in particular, every simple braiding (which is a positive braid) is conjugate to each other. Using this, we get

Proposition 5.15 (see e.g. [FuS])**.** *The abelianization of the group B_∞ is $H_1(B_\infty) = \mathbb{Z}$.*

So the abelianization homomorphism $B_\infty \to H_1(B_\infty) = \mathbb{Z}$ induces the central extensions $T^*_{ab} \cong T^*/[B_\infty, T^*]$ and $T^\sharp_{ab} \cong T^\sharp/[B_\infty, T^\sharp]$ of T (where we denote the subgroups of T^* and $T^\sharp$ isomorphic to B_∞ as mentioned in Prop. 5.12 by B_∞ here, by abuse of notation), as in the diagrams below:

$$\begin{tikzcd} 1 \arrow[r] & B_\infty \arrow[r] \arrow[d] & T^* \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \mathbb{Z} \arrow[r] & T^*_{ab} \arrow[r] & T \arrow[r] & 1, \end{tikzcd} \tag{5.5}$$

and

$$\begin{tikzcd} 1 \arrow[r] & B_\infty \arrow[r] \arrow[d] & T^\sharp \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, "id"] & 1 \\ 1 \arrow[r] & \mathbb{Z} \arrow[r] & T^\sharp_{ab} \arrow[r] & T \arrow[r] & 1. \end{tikzcd} \tag{5.6}$$

Notation 5.16. *For $\diamondsuit \in \{*, \sharp\}$, we denote by $\widetilde{\alpha}^\diamondsuit$, $\widetilde{\beta}^\diamondsuit \in T^\diamondsuit_{ab}$ the images of $\alpha^\diamondsuit$, $\beta^\diamondsuit \in T^\diamondsuit$.*

Proposition 5.17 ([FuKap2])**.** *The group T^*_{ab} has the presentation with three generators $\widetilde{\alpha}^*, \widetilde{\beta}^*$, and z and the relations*

$$\begin{aligned} &(\widetilde{\beta}^*\widetilde{\alpha}^*)^5 = z, \qquad (\widetilde{\alpha}^*)^4 = 1, \qquad (\widetilde{\beta}^*)^3 = 1, \qquad [\widetilde{\alpha}^*, z] = \left[\widetilde{\beta}^*, z\right] = 1, \\ &\left[\widetilde{\beta}^*\widetilde{\alpha}^*\widetilde{\beta}^*, (\widetilde{\alpha}^*)^2\widetilde{\beta}^*\widetilde{\alpha}^*\widetilde{\beta}^*(\widetilde{\alpha}^*)^2\right] = \left[\widetilde{\beta}^*\widetilde{\alpha}^*\widetilde{\beta}^*, (\widetilde{\alpha}^*)^2\widetilde{\beta}^*(\widetilde{\alpha}^*)^2\widetilde{\beta}^*\widetilde{\alpha}^*\widetilde{\beta}^*(\widetilde{\alpha}^*)^2(\widetilde{\beta}^*)^2(\widetilde{\alpha}^*)^2\right] = 1, \end{aligned} \tag{5.7}$$

and the projection map $T^*_{ab} \to T$ (in (5.5)) sends $\widetilde{\alpha}^*$ to α and $\widetilde{\beta}^*$ to β. Hence

$$T^*_{ab} \cong T_{1,0,0,0} \tag{5.8}$$

where $T_{n,p,q,r}$ is as in Thm. 4.7.

The group $T^\sharp_{ab}$ has the presentation with the three generators $\widetilde{\alpha}^\sharp, \widetilde{\beta}^\sharp$, and z and the relations

$$\begin{aligned} &(\widetilde{\beta}^\sharp\widetilde{\alpha}^\sharp)^5 = z^3, \qquad (\widetilde{\alpha}^\sharp)^4 = z^2, \qquad (\widetilde{\beta}^\sharp)^3 = 1, \qquad [\widetilde{\alpha}^\sharp, z] = \left[\widetilde{\beta}^\sharp, z\right] = 1, \\ &\left[\widetilde{\beta}^\sharp\widetilde{\alpha}^\sharp\widetilde{\beta}^\sharp, (\widetilde{\alpha}^\sharp)^2\widetilde{\beta}^\sharp\widetilde{\alpha}^\sharp\widetilde{\beta}^\sharp(\widetilde{\alpha}^\sharp)^2\right] = \left[\widetilde{\beta}^\sharp\widetilde{\alpha}^\sharp\widetilde{\beta}^\sharp, (\widetilde{\alpha}^\sharp)^2\widetilde{\beta}^\sharp(\widetilde{\alpha}^\sharp)^2\widetilde{\beta}^\sharp\widetilde{\alpha}^\sharp\widetilde{\beta}^\sharp(\widetilde{\alpha}^\sharp)^2(\widetilde{\beta}^\sharp)^2(\widetilde{\alpha}^\sharp)^2\right] = 1, \end{aligned} \tag{5.9}$$

where the projection map $T^\sharp_{ab} \to T$ (in (5.6)) sends $\widetilde{\alpha}^\sharp$ to α and $\widetilde{\beta}^\sharp$ to β. Therefore we have

$$T^\sharp_{ab} \cong T_{3,2,0,0}. \tag{5.10}$$

Corollary 5.18. *The isomorphisms (5.8), (5.10), together with Theorems 4.8 and 4.9, imply*

$$\widehat{G}^{CF}_{mark} \cong T^*_{ab}, \tag{5.11}$$

$$\widehat{G}^{Kash}_{mark} \cong T^\sharp_{ab}. \tag{5.12}$$

It is exactly from Thm. 4.8 and (5.8) that Funar and Sergiescu concluded (5.11). Although Thm. 4.9 proved in the present paper, together with (5.10), implies (5.12), we shall give a direct topological proof of (5.12) in the following subsections, not depending on (5.10). Then, using the result (5.10) of Funar-Kapoudjian (implied by [FuKap2]), we can recover Thm. 4.9 (and we view this as a 'topological proof' of Thm. 4.9).

5.3. **Dualizing between the ribbon graphs and the tessellations of $\mathbb{D}$.** For this topological proof, instead of the ribbon graphs (the fat infinite binary trees) we use its dual picture, namely the (Farey-type) tessellations (of the unit disc $\mathbb{D}$) studied in §2. These two approaches are equivalent, as we shall soon see. For formulating the result of the present paper about the 'dotted $\sharp$-punctured tessellations on $\mathbb{D}^\sharp$' in terms of the ribbon graph language, readers can consult e.g. Rem.5.22.

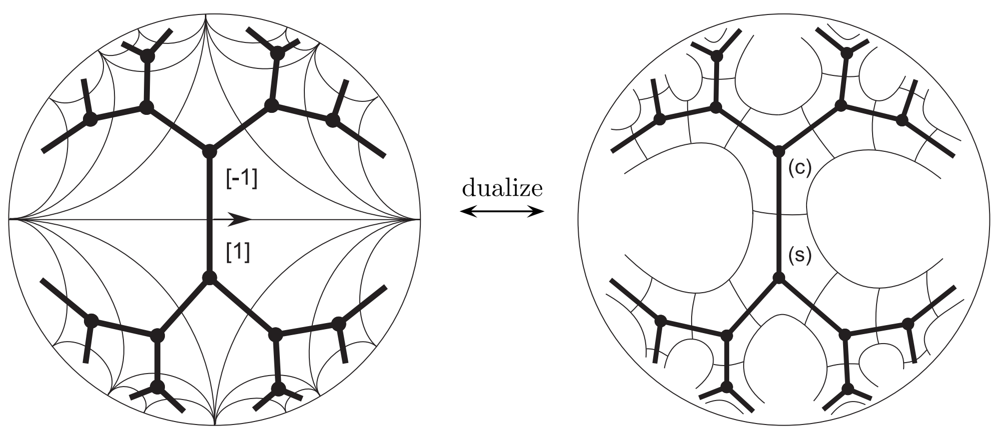


FIGURE 22. Dualizing a (marked) tessellation to a ribbon graph

The 'dualizing' process is as depicted in Fig. 22, which is quite straightforward. By this dualizing, we get a correspondence between $\mathcal{F}tess$ (Def. 2.7) and the isotopy classes of rigid structures on the ribbon tree D (Def. 5.3), as well as that between $\mathcal{F}tess_{mark}$ (Def. 2.8) and the isotopy classes of marked rigid structures on the ribbon tree D (Def. 5.5), by matching the triangles $[-1], [1]$ with the hexagons $(c), (s)$, respectively, as in Fig. (22).

Note that by this dualizing, we have the following correspondence:

$$\text{interior of a triangle} \longleftrightarrow \text{interior of a hexagon,} \tag{5.13}$$

$$\text{three edges of a triangle} \longleftrightarrow \text{three separating arcs of a hexagon,} \tag{5.14}$$

$$\text{three vertices of a triangle} \longleftrightarrow \text{three non-separating arcs of a hexagon,} \tag{5.15}$$

where the 'separating arcs' of a hexagon are the sides of the hexagon traversing the interior of the ribbon tree, while the 'non-separiting arcs' are the ones which constitute the 'frontier' (boundary) of the ribbon tree (see Def. 5.4 for 'frontier'). In fact, if a ribbon tree is taken to be an open set (i.e. not containing any boundary point), then this 'dualizing' yields a homeomorphism from $\mathbb{D}$ to D, taking a tessellation to a rigid structure on D (and taking the (decorated) standard tessellations the (decorated) canonical rigid structures).

Since we can have punctures either on the separating arcs (D^*) or in the interior of the hexagons ($D^\sharp$) in the ribbon tree picture, we transfer this idea of puncturing (via the above mentioned correspondence, or even a homeomorphism) to the world of tessellations, so that for a tessellation we allow punctures either on the ideal arcs or in the interior of the ideal triangles:

Definition 5.19. *Denote by $\mathbb{D}^*$ (resp. $\mathbb{D}^\sharp$) the open unit disc $\mathbb{D}$ with infinitely many punctures (depicted as $\circ$ in the pictures, to avoid the confusion with the dots $\bullet$ for dotted tessellations), where the position of the punctures are chosen once and for all, described as follows. Choose one point in the interior of each ideal arc (resp. one point in the interior of each ideal triangle) of the Farey tessellation (see Def. 2.5 for the Farey tessellation) where we assume here that all the ideal arcs are stretched to geodesics; these points comprise the punctures for $\mathbb{D}^*$ (resp. for $\mathbb{D}^\sharp$). We call the punctures of $\mathbb{D}^*$ (resp. $\mathbb{D}^\sharp$) the* $*$-punctures *(resp.* $\sharp$-punctures*).*

An ideal arc *in $\mathbb{D}^*$ (resp. in $\mathbb{D}^\sharp$) connecting two given distinct (rational) points on $S^1 = \partial\mathbb{D}$ is a homotopy class of paths (with no orientation) connecting the two points (one can think of the paths being in the closure $\overline{\mathbb{D}}$), while the homotopy requires that each ideal arc contains exactly one $*$-puncture at all times (resp. that each ideal arc does not pass through any $\sharp$-puncture at any time). An* ideal triangle *in $\mathbb{D}^*$ (resp. in $\mathbb{D}^\sharp$) is a triangle with three distinct vertices whose sides are ideal arcs of $\mathbb{D}^*$ (resp. of $\mathbb{D}^\sharp$).*

A $*$-Farey ideal arc *(resp.* $\sharp$-Farey ideal arc*) is the homotopy class of ideal arcs in $\mathbb{D}^*$ (resp. in $\mathbb{D}^\sharp$) homotopic to an ideal arc of the Farey tesssellation of $\mathbb{D}$ (Def. 2.5) stretched to the hyperbolic geodesic.*

A (Farey-type) $*$-punctured tessellation of $\mathbb{D}^*$ *(resp.* (Farey-type) $\sharp$-punctured tessellation of $\mathbb{D}^\sharp$*) is a (Farey-type) tessellation of $\mathbb{D}$ (Def. 2.7) such that each ideal arc goes through exactly one $*$-puncture of $\mathbb{D}^*$ while every $*$-puncture is being passed by one arc (resp. such that each ideal triangle contains in its interior exactly one $\sharp$-puncture), and such that all but finitely many ideal arcs are $*$-Farey ideal arcs (resp. $\sharp$-Farey ideal arcs). Each ideal arc should now be thought of as an ideal arc in $\mathbb{D}^*$ (resp. in $\mathbb{D}^\sharp$), in the sense as described above.*

Let $\diamondsuit = *$ or $\sharp$. A marked $\diamondsuit$-punctured tessellation of $\mathbb{D}^\diamondsuit$ *is a $\diamondsuit$-punctured tessellation of $\mathbb{D}^\diamondsuit$ together with the choice of a distinguished oriented arc (sometimes written d.o.e., standing for distinguished oriented edge), just like in the definition of marked tessellation in Def. 2.8.*

A dotted $\diamondsuit$-punctured tessellation of $\mathbb{D}^\diamondsuit$ *is a $\diamondsuit$-punctured tessellation of $\mathbb{D}^\diamondsuit$ together with the choice of a distinguished corner for each ideal triangle denoted by a filled dot $\bullet$ in the pictures, and the choice of labeling for ideal triangles by $\mathbb{Q}^\times$ (i.e. a bijection between the ideal triangles and $\mathbb{Q}^\times$, where the triangle labeled by $j \in \mathbb{Q}^\times$ will be denoted by $[j]$ in the picture), just like in the definition of dotted tessellation in Def. 2.9.*

For $\diamondsuit \in \{*, \sharp\}$, we denote by

$$\mathcal{F}tess^\diamondsuit, \quad \mathcal{F}tess^\diamondsuit_{mark}, \quad \text{and} \quad \mathcal{F}tess^\diamondsuit_{dot}, \tag{5.16}$$

the set of all $\diamondsuit$-punctured tessellations of $\mathbb{D}^\diamondsuit$, the set of all marked $\diamondsuit$-punctured tessellations of $\mathbb{D}^\diamondsuit$, and the set of all dotted $\diamondsuit$-punctured tessellations of $\mathbb{D}^\diamondsuit$, respectively.

Remark 5.20. *Due to the restriction of the homotopy used in the definition of ideal arcs, now there are infinitely many distinct ideal arcs connecting given two distinct points on $S^1 = \partial\mathbb{D}$ (for the non-punctured case there was a unique ideal arc connecting any given two distinct points).*

Remark 5.21. *The '$*$' appearing in the notation for standard tessellation τ^*, standard marked (resp. dotted) tessellations τ^*_{mark} (resp. τ^*_{dot}) can create a confusion with the '$*$' in the $*$-puncturing. Therefore we refrained from giving a notation for standard elements of $\mathcal{F}tess^\diamondsuit$, $\mathcal{F}tess^\diamondsuit_{mark}$, and $\mathcal{F}tess^\diamondsuit_{dot}$ for the moment.*

Remark 5.22. *We already have the ribbon graph counterparts for the $\diamondsuit$-punctured tessellations and the marked $\diamondsuit$-punctured tessellations on $\mathbb{D}^\diamondsuit$ (the choice of a d.o.e. is equivalent to the choice of hexagons (c) and (s) in ribbon graph; one can even think of recording this as 'distinguished oriented separating arc' on the ribbon graph). We can also consider the ribbon graph counterparts for the dotted $\diamondsuit$-punctured tessellations on $\mathbb{D}^\diamondsuit$, by drawing a dot $\bullet$ on a non-separating arc of each hexagon, as shown in Fig. 23.*

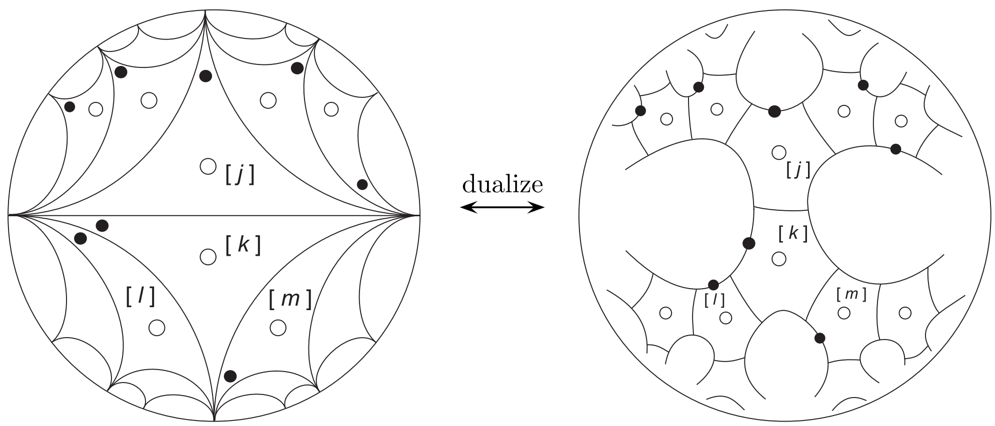


FIGURE 23. Dualizing a dotted $\sharp$-punctured tessellation to a ribbon graph

Just as in the non-punctured case, we have the following two natural maps

$$\mathcal{F}tess^\diamondsuit_{mark} \to \mathcal{F}tess^\diamondsuit, \quad \mathcal{F}tess^\diamondsuit_{dot} \to \mathcal{F}tess^\diamondsuit \tag{5.17}$$

which forget the decorations and return the underlying $\diamondsuit$-punctured tessellation of $\mathbb{D}^\diamondsuit$.

Definition 5.23. *For $\diamondsuit \in \{*, \sharp\}$, define the mappings*

$$F^\diamondsuit : \mathcal{F}tess^\diamondsuit_{mark} \to \mathcal{F}tess^\diamondsuit_{dot} \tag{5.18}$$

exactly as in Def. 2.11 (in particular, this map does not change the underlying $\diamondsuit$-punctured tessellation).

One can easily observe the following.

Proposition 5.24. *The map $F^\diamondsuit$ is injective.*

Of course, we also have the following maps

$$\mathcal{F}tess^\diamondsuit \to \mathcal{F}tess, \quad \mathcal{F}tess^\diamondsuit_{mark} \to \mathcal{F}tess_{mark}, \quad \mathcal{F}tess^\diamondsuit_{dot} \to \mathcal{F}tess_{dot}, \tag{5.19}$$

which forget the punctures.

We can now define an automorphism group of $\mathcal{F}tess^\diamondsuit_{mark}$, mimicking the non-punctured case. We do this only for $\diamondsuit = \sharp$ in the present paper. Recall that in §2.2 the generating automorphisms

α and β of $\mathcal{F}tess_{mark}$ were first defined as a combinatorial change of marked tessellations in Def. 2.17, and then interpreted as induced from asymptotically rigid homeomorphisms of $\mathbb{D}$ in Prop. 2.19. Similarly, we first define the automorphisms $\underline{\alpha}^\sharp$, $\underline{\beta}^\sharp$ and $\underline{\sigma}$ of $\mathcal{F}tess^\sharp_{mark}$ combinatorially, and consider the group generated by them:

Definition 5.25. *Let $G^\sharp_{mark}$ be the group of automorphisms of $\mathcal{F}tess^\sharp_{mark}$ generated by the automorphisms $\underline{\alpha}^\sharp$, $\underline{\beta}^\sharp$ and $\underline{\sigma}$:*

$$G^\sharp_{mark} = \langle \underline{\alpha}^\sharp, \underline{\beta}^\sharp, \underline{\sigma} \rangle, \tag{5.20}$$

where $\underline{\alpha}^\sharp$ and $\underline{\beta}^\sharp$ are defined as seen in Figures 24 and 25, leaving the other part (denoted by the triple dots '$\cdots$') intact; similarly as for the actions of α and β on $\mathcal{F}tess_{mark}$ (Def. 2.17), one can think that the $\underline{\alpha}^\sharp$ action rotates the d.o.e. counterclockwise to the other diagonal without passing through the $\sharp$-punctures, and the $\underline{\beta}^\sharp$ action just alters the choice of d.o.e.

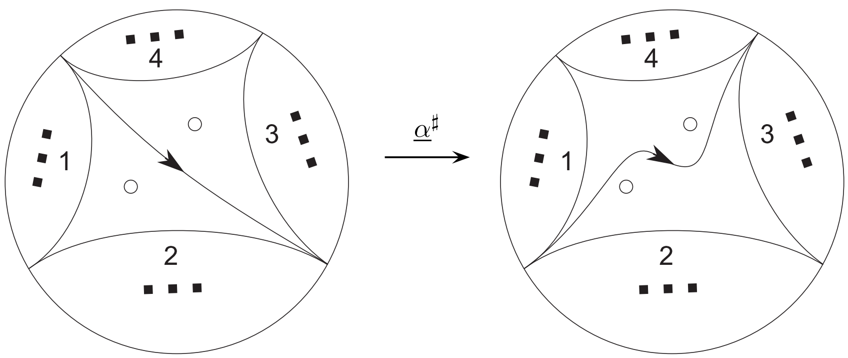


Figure 24. The action of $\underline{\alpha}^\sharp$ on $\mathcal{F}tess^\sharp_{mark}$

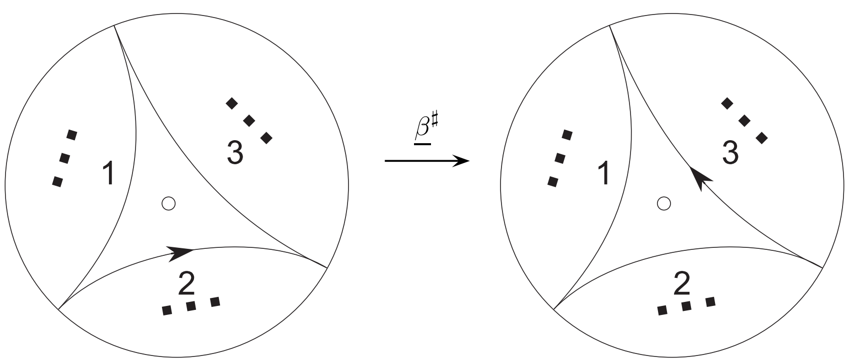


Figure 25. The action of $\underline{\beta}^\sharp$ on $\mathcal{F}tess^\sharp_{mark}$

The action of $\underline{\sigma}$ on any marked $\sharp$-punctured tessellation is as induced from the braiding homeomorphism (see Def. 5.9) associated to an arc e connecting the two $\sharp$-punctures of the two ideal

triangles in $\mathbb{D}^\sharp$ (Def. 5.19) traversing exactly one ideal arc (which can be called a simple *arc as in Def. 5.10).*

As done in Prop. 2.19 for G_{mark}, one may want to characterize the elements of $G^\sharp_{mark}$ as the automorphisms of $\mathcal{F}tess^\sharp_{mark}$ induced by certain homeomorphisms of $\mathbb{D}^\sharp$. Let's first consider the case of $\underline{\alpha}^\sharp$. The homeomorphism (class) which would induce the $\underline{\alpha}^\sharp$ action by acting on a marked $\sharp$-punctured tessellation would be expected to rotate the picture by the angle $\frac{\pi}{2}$ roughly speaking, hence permuting the four branches denoted by $1, 2, 3, 4$ (with the triple dots) in Fig. 24 issued from the ideal quadrilateral containing the d.o.e. as a diagonal; let us call this ideal quadrilateral 'the central quadrilateral' for the moment. In this sense, the labeling of the four branches in the RHS of Fig. 24 may look misleading. Indeed, we want the homeomorphism to take the part labeled 1 to the part labeled 2, and the part labeled 2 to the part labeled 3, etc.

Nevertheless, for each $j = 1, 2, 3, 4$, we do want that the branch labeled by j in the LHS of Fig. 24 and the branch j in the RHS are exactly same, i.e. have the same ideal arcs. And then we would look for a homeomorphism (class) φ_α of $\mathbb{D}^\sharp$ which rotates the central quadrilateral 'by the angle $\frac{\pi}{2}$', therefore in particular permuting the four vertices of the quadrilateral cyclically counterclockwise, while fixing the two punctures in the quadrilateral, and which 'maps the part j to the part $j+1$' (the labels for the four branches are viewed cyclically), in the following sense. Let $\tau^\sharp_{mark}$ be the relevant marked $\sharp$-punctured tessellation, on which this sought-for homeomorphism (class) would act on by the action $\underline{\alpha}^\sharp$. First, we require that φ_α induces a bijection from the set of all vertices (i.e. the endpoints of the ideal arcs in $\tau^\sharp_{mark}$) in the part j including the ones at the two extreme (so these two are vertices of the central quadrilateral) to the set of all vertices in the part $j+1$. Second, we require that the vertices $x, y \in S^1$ in the part j are connected by an ideal arc of $\tau^\sharp_{mark}$ if and only if their images under φ_α in part $j+1$ are connected by an ideal arc of $\tau^\sharp_{mark}$; this requirement (together with the images under φ_α of the two extreme vertices in the part j) settles the image under φ_α of all the vertices in the part j. Third, if $x, y \in S^1$ in the part j are connected by an ideal arc, then we require that φ_α takes the ideal arc of $\tau^\sharp_{mark}$ connecting x, y to the ideal arc of $\tau^\sharp_{mark}$ connecting $\varphi_\alpha(x), \varphi_\alpha(y)$. Such a homeomorphism φ_α of $\mathbb{D}^\sharp$ (i.e. the open unit disc $\mathbb{D}$ minus the $\sharp$-punctures) is uniquely determined up to isotopy of $\mathbb{D}^\sharp$ fixing the rational boundary points on $S^1 = \partial\mathbb{D}$ (and fixing the $\sharp$-punctures). We can find a similar homeomorphism (class) φ_β of $\mathbb{D}^\sharp$ for the $\underline{\beta}^\sharp$ action, by requiring it to rotate the 'central' ideal triangle counterclockwise and taking each ideal arc of $\tau^\sharp_{mark}$ in the part j to the 'corresponding' ideal arc of $\tau^\sharp_{mark}$ in the part $j+1$.

We can characterize the (isotopy classes of the) homeomorphisms which induce the elements of $G^\sharp_{mark}$, as we did in Prop. 2.19. First, we need an analog for $\mathbb{D}^\sharp$ of the asymptotically rigid homeomorphisms of $\mathbb{D}$ (see Def. 2.13):

Definition 5.26. *Regard $\mathbb{D}^\sharp$ as the open unit disc minus the $\sharp$-punctures (Def. 5.19). Denote the isotopy of $\mathbb{D}^\sharp$ fixing all the rational points on the boundary $S^1 = \partial\mathbb{D}$ by the* $\sharp$-isotopy. *A $\sharp$-isotopy class of homeomorphisms of $\mathbb{D}^\sharp$ to itself, which are allowed to permute the $\sharp$-punctures, is said to be* asymptotically $\sharp$-rigid *if its continuous extension to the boundary S^1 is a $PPSL(2,\mathbb{Z})$ homeomorphism of S^1 (see Def. 2.13), and takes all but finitely many $\sharp$-Farey ideal arcs to $\sharp$-Farey ideal arcs (see Def. 5.19 for the $\sharp$-Farey ideal arcs).*

Then, we can easily see that each of the actions of $\underline{\alpha}^\sharp, \underline{\beta}^\sharp, \underline{\sigma}$, and any element of $G^\sharp_{mark}$ on a given marked $\sharp$-punctured tessellation is induced by (a $\sharp$-isotopy class of) an asymptotically $\sharp$-rigid homeomorphism of $\mathbb{D}^\sharp$. In fact, we can say the following:

Proposition 5.27. *The action of any element of $G^\sharp_{mark}$ on a given marked $\sharp$-punctured tessellation is induced by a $\sharp$-isotopy class of asymptotically $\sharp$-rigid homeomorphisms of $\mathbb{D}^\sharp$. Moreover, $G^\sharp_{mark}$ is the group of all automorphisms of $\mathcal{F}tess^\sharp_{mark}$ induced by asymptotically $\sharp$-rigid homeomorphisms.*

Instead of trying to prove the second assertion of the above proposition, we will just use the result of [FuKap2] as stated in Thm. 5.13 in the present paper, which says that the asymptotically rigid mapping class group $T^\sharp$ of the ($\sharp$-punctured) ribbon tree $D^\sharp$ is generated by $\alpha^\sharp$, $\beta^\sharp$ and σ. We mentioned earlier in the current subsection that there is a homeomorphism from the unit disc $\mathbb{D}$ to the ribbon tree D, taking a tessellation to a rigid structure on D. We can also have such a homeomorphism from $\mathbb{D}^\sharp$ to $D^\sharp$, that takes the 'standard' $\sharp$-tessellation of $\mathbb{D}^\sharp$ (i.e. consisting of all the $\sharp$-Farey ideal arcs; see Def. 5.19) to the canonical rigid structure of $D^\sharp$. It is not too difficult to see that under this homeomorphism, the homeomorphisms of $\mathbb{D}^\sharp$ inducing $\underline{\alpha}^\sharp$, $\underline{\beta}^\sharp$ (discussed right after Def. 5.25) and $\underline{\sigma}$ correspond to the homeomorphisms $\alpha^\sharp$, $\beta^\sharp$ and σ of $D^\sharp$, and the asymptotically $\sharp$-rigid homeomorphisms of $\mathbb{D}^\sharp$ (Def. 5.26) correspond to the asymptotically rigid homeomorphisms of $D^\sharp$ (Def. 5.4):

Proposition 5.28. *The above formulation of $G^\sharp_{mark}$ is essentially equivalent to that of $T^\sharp$ (via the homeomorphism between $\mathbb{D}^\sharp$ and $D^\sharp$). More explicitly, we have the following isomorphism*

$$T^\sharp \xrightarrow{\sim} G^\sharp_{mark} : \alpha^\sharp \mapsto \underline{\alpha}^\sharp, \quad \beta^\sharp \mapsto \underline{\beta}^\sharp, \quad \sigma \mapsto \underline{\sigma} \tag{5.21}$$

(see §5.1 for $\alpha^\sharp$, $\beta^\sharp$ and σ, and Thm. 5.13 to see that $T^\sharp$ is generated by these three generators).

Thus we can regard $G^\sharp_{mark}$ as the 'asymptotically $\sharp$-rigid' mapping class group of $\mathbb{D}^\sharp$, and therefore it also contains a subgroup isomorphic to the infinite braid group B_∞ (see Def. 5.10 and 5.11, and Prop. 5.12), which can be defined as the subgroup of $G^\sharp_{mark}$ generated by the elements induced by the braidings for the simple edges (i.e. traversing exactly one ideal arc and only once) between two distinct punctures; see Def. 5.9 for the braiding homeomorphisms.

Remark 5.29. *The group G^*_{mark} of automorphisms of $\mathcal{F}tess^*_{mark}$ can be defined and compared to T^* in an analogous manner, through the appropriate homeomorphism between $\mathbb{D}^*$ and D^*. We note here that T^* is proved to be generated by α^* and β^*, while $T^\sharp$ is not proved to be generated by $\alpha^\sharp$ and $\beta^\sharp$. This is why we put $\underline{\sigma}$ in the generators of $G^\sharp_{mark}$ in Def. 5.25.*

Just as for $T^\sharp$ in the ribbon graph case, forgetting punctures yield the homomorphism

$$G^\sharp_{mark} \longrightarrow G_{mark} : \underline{\alpha}^\sharp \mapsto \alpha, \quad \underline{\beta}^\sharp \mapsto \beta, \quad \underline{\sigma} \mapsto 1 \tag{5.22}$$

with the kernel B_∞. Thus we get a short exact sequence $1 \to B_\infty \to G^\sharp_{mark} \to G_{mark} \to 1$. Now, the abelianization homomorphism $B_\infty \to H_1(B_\infty) = \mathbb{Z}$ induces the following commutative diagram for the relative abelianization $(G^\sharp_{mark})_{ab} \cong G^\sharp_{mark}/[B_\infty, G^\sharp_{mark}]$, which is a central extension of G_{mark} by $\mathbb{Z}$:

$$\begin{tikzcd} 1 \arrow[r] & B_\infty \arrow[r] \arrow[d] & G^\sharp_{mark} \arrow[r] \arrow[d] & G_{mark} \arrow[r] \arrow[d, "id"] & 1, \\ 1 \arrow[r] & \mathbb{Z} \arrow[r] & (G^\sharp_{mark})_{ab} \arrow[r] & G_{mark} \arrow[r] & 1. \end{tikzcd} \tag{5.23}$$

Via the identification $T \cong G_{mark}$ (see Prop. 5.7), the above diagram is equivalent to (5.6), with 'G_{mark}' replaced by 'T'. Meanwhile, e.g. from the presentation of $T^\sharp_{ab}$ in [FuKap2], we have:

Proposition 5.30. *The group $T^\sharp_{ab}$ is generated by $\widetilde{\alpha}^\sharp$ and $\widetilde{\beta}^\sharp$ (see Notation 5.16).*

Corollary 5.31. *Denote the images of* $\underline{\alpha}^\sharp$ *and* $\underline{\beta}^\sharp$ *under the homomorphism* $G^\sharp_{mark} \to (G^\sharp_{mark})_{ab}$ *by* $\widetilde{\underline{\alpha}}^\sharp$ *and* $\widetilde{\underline{\beta}}^\sharp$*, respectively. Then we have the following isomorphism*

$$T^\sharp_{ab} \xrightarrow{\sim} (G^\sharp_{mark})_{ab} : \widetilde{\alpha}^\sharp \mapsto \widetilde{\underline{\alpha}}^\sharp, \quad \widetilde{\beta}^\sharp \mapsto \widetilde{\underline{\beta}}^\sharp \tag{5.24}$$

(see Notation 5.16). So the projection map $(G^\sharp_{mark})_{ab} \to G_{mark}$ *in* (5.23) *can be written as*

$$(G^\sharp_{mark})_{ab} \to G_{mark} : \widetilde{\underline{\alpha}}^\sharp \mapsto \alpha, \quad \widetilde{\underline{\beta}}^\sharp \mapsto \beta. \tag{5.25}$$

So now, proving (5.12) is equivalent to proving

$$\widehat{G}^{Kash}_{mark} \cong (G^\sharp_{mark})_{ab}. \tag{5.26}$$

For this, we need to study the $\sharp$-punctured version of the Kashaev group G_{dot} (Def. 2.30), which can be thought of as generated by the elementary changes of dotted tessellations. This will be done in the following subsection.

5.4. **The $\sharp$-punctured version $G^\sharp_{dot}$ of the Kashaev group G_{dot}.** As in §2.2, instead of thinking of a group of automorphisms of $\mathcal{F}tess^\sharp_{dot}$, it is more natural to start with a groupoid (which can e.g. be denoted by $Pt^\sharp_{dot}$) whose set of objects is $\mathcal{F}tess^\sharp_{dot}$, such that for each two objects there is exactly one morphism. As done in Def. 2.27, we describe the elementary moves of dotted $\sharp$-punctured tessellations, each of which represents some class of morphisms of this groupoid:

Definition 5.32. *We describe the* elementary moves $A^\sharp_{[j]}$, $T^\sharp_{[j][k]}$, $P^\sharp_{(jk)}$ *and* $\sigma_{[j][k]}$ *of* $\mathcal{F}tess^\sharp_{dot}$*, for* $j,k \in \mathbb{Q}^\times$ *(triangle labels, where* $j \neq k$*).*

1) *The move* $A^\sharp_{[j]}$ *acts on any dotted $\sharp$-punctured tessellation by moving the dot* $\bullet$ *(i.e. distinguished corner) of the triangle labeled by* $j \in \mathbb{Q}^\times$ *counterclockwise to the next corner in that triangle, while leaving all other information intact. See Fig. 26.*

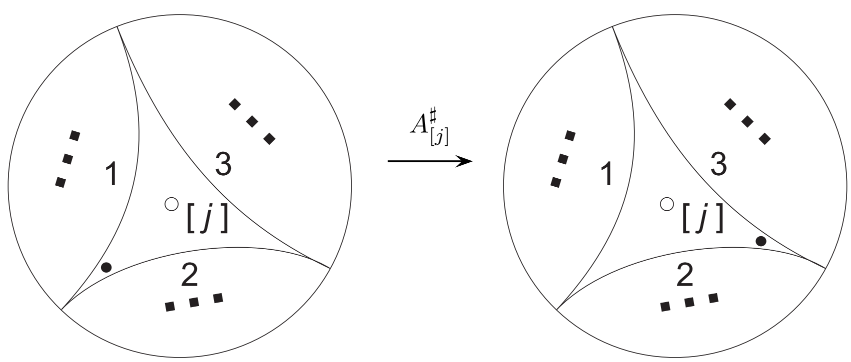


FIGURE 26. The action of $A^\sharp_{[j]}$ on $\mathcal{F}tess^\sharp_{dot}$

2) *The move* $T^\sharp_{[j][k]}$ *acts on a dotted $\sharp$-punctured tessellation only if the triangles labeled by* j *and* k *are adjacent to each other and the dots* $\bullet$ *of them are configured precisely as in Fig. 27 (relative to the common arc of the two triangles); the $\sharp$-punctures can be wrapped around by the arcs in any way. Just as for* $T_{[j][k]}$*, the action* $T^\sharp_{[j][k]}$ *replaces*

the common arc of the two triangles by the other diagonal arc of the ideal quadrilateral formed by those two triangles, by rotating this arc clockwise *while letting the dots* $\bullet$ *and the triangle labels be 'floating' and thus pushed accordingly by the rotating arc. This 'rotating' can be thought of as a homotopy from the previous diagonal arc to the new diagonal arc by sliding the two endpoints along the edges of the above mentioned quadrilateral in the* clockwise *direction, while requiring the arc not to pass through any* $\sharp$*-puncture at any time. This elementary move leaves all the other information intact. See Fig. 27.*

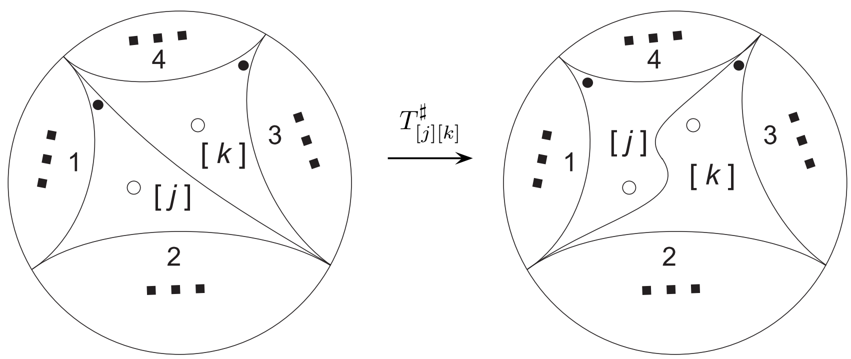


FIGURE 27. The action of $T^\sharp_{[j][k]}$ on $\mathcal{F}tess^\sharp_{dot}$

3) *The move* $P^\sharp_{(jk)}$ *acts by exchanging the labels of the triangles labeled by* j *and* k*, and leave all other information intact. More generally, for a permutation* γ *of* $\mathbb{Q}^\times$*, the move* $P^\sharp_\gamma$ *acts by replacing the label* j *for each triangle by* $\gamma(j)$*, and leaves all other information intact.*
4) *The move* $\sigma_{[j][k]}$ *acts on a dotted* $\sharp$*-punctured tessellation only if the triangles labeled by* j *and* k *are adjacent to each other. Its action is as induced by the braiding homeomorphism associated to a simple edge (i.e. traversing exactly one ideal arc and only once) connecting the punctures of the triangles* j *and* k*; see Def. 5.9 for braidings.*

Proposition 5.33. *The above elementary moves satisfy the following relations:*

$$(A^\sharp_{[j]})^3 = id, \tag{5.27}$$

$$T^\sharp_{[k][\ell]} T^\sharp_{[j][k]} = T^\sharp_{[j][k]} T^\sharp_{[j][\ell]} T^\sharp_{[k][\ell]}, \tag{5.28}$$

$$A^\sharp_{[j]} T^\sharp_{[j][k]} A^\sharp_{[k]} = A^\sharp_{[k]} T^\sharp_{[k][j]} A^\sharp_{[j]}, \tag{5.29}$$

$$T^\sharp_{[j][k]} A^\sharp_{[j]} T^\sharp_{[k][j]} = \sigma^{-1}_{[j][k]} A^\sharp_{[j]} A^\sharp_{[k]} P^\sharp_{(jk)}, \tag{5.30}$$

where $j,k,\ell \in \mathbb{Q}^\times$ *are mutually distinct. Also, the trivial relations for index permutations* $P^\sharp_{(jk)}$ *hold precisely as in* (2.15)*, and we have the braid relations between the braid generators. Any two words in the elementary moves whose collections of subscripts (indices) don't intersect with each other commute.*

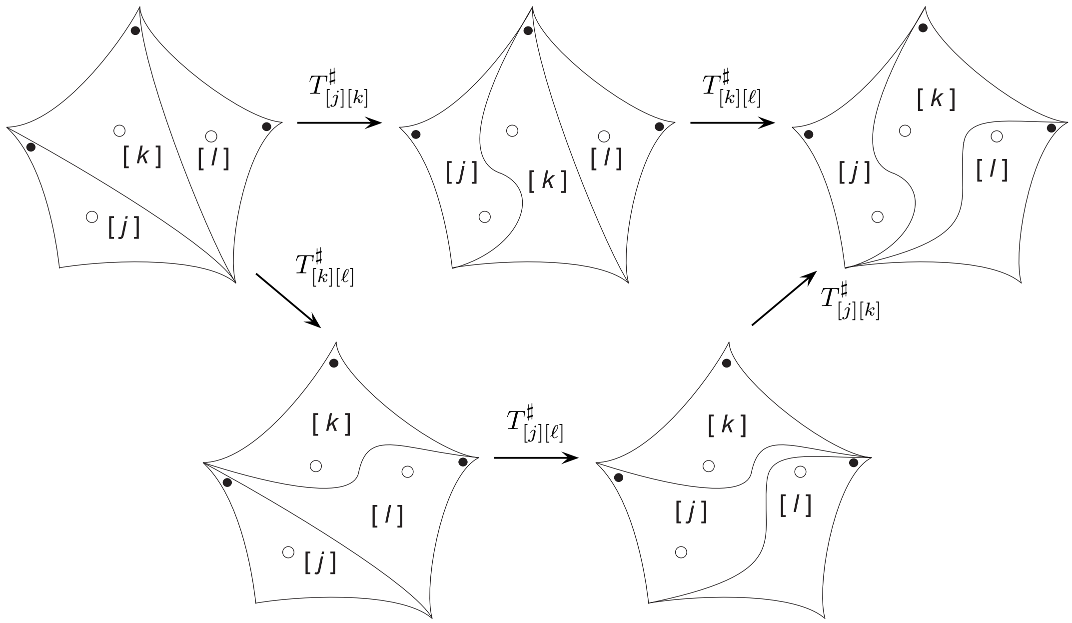


FIGURE 28. The pentagon relation $T^\sharp_{[k][\ell]} T^\sharp_{[j][k]} = T^\sharp_{[j][k]} T^\sharp_{[j][\ell]} T^\sharp_{[k][\ell]}$ on $\mathcal{F}tess^\sharp_{dot}$

Proof. The proof of (5.27), the trivial relations of index permutations, and the braid relations is immediate. The proof of (5.28), (5.29) and (5.30) is manifest from Figures 28, 29 and 30, respectively.

□

Definition 5.34. *Let $G^\sharp_{dot}$ be the group presented by the generators $A^\sharp_{[j]}$, $T^\sharp_{[j][k]}$, $P^\sharp_{(jk)}$, $\sigma_{[j][k]}$ for $j,k \in \mathbb{Q}^\times$ ($j \neq k$) and relations satisfied by the corresponding elementary moves of $\mathcal{F}tess^\sharp_{dot}$.*

Remark 5.35. *There may be some relations not generated by those shown in Prop. 5.33.*

Remark 5.36. *As in Rem. 2.31, to be more precise, we would want to include the more general index permutations $P^\sharp_\gamma$ (for permutation γ of $\mathbb{Q}^\times$) in the group $G^\sharp_{dot}$.*

It is easy to see that $A^\sharp_{[j]}$, $T^\sharp_{[j][k]}$, $P^\sharp_{(jk)}$ generate $G^\sharp_{dot}$ (by (5.30)). By forgetting the punctures we get the group homomorphism

$$G^\sharp_{dot} \to G_{dot} : A^\sharp_{[j]} \mapsto A_{[j]}, \quad T^\sharp_{[j][k]} \to T_{[j][k]}, \quad P^\sharp_{(jk)} \to P_{(jk)}, \tag{5.31}$$

with the kernel isomorphic to B_∞ (see Def. 5.10 and 5.11, and also the remarks following Prop. 5.28), the braid group generated by $\sigma_{[j][k]}$ (Def. 5.32), thus yielding the exact sequence

$$1 \longrightarrow B_\infty \longrightarrow G^\sharp_{dot} \longrightarrow G_{dot} \longrightarrow 1. \tag{5.32}$$

The abelianization homomorphism $B_\infty \to H_1(B_\infty) = \mathbb{Z}$ induces the following commutative diagram for the relative abelianization $(G^\sharp_{dot})_{ab} \cong G^\sharp_{dot}/[B_\infty, G^\sharp_{dot}]$, which is a central extension

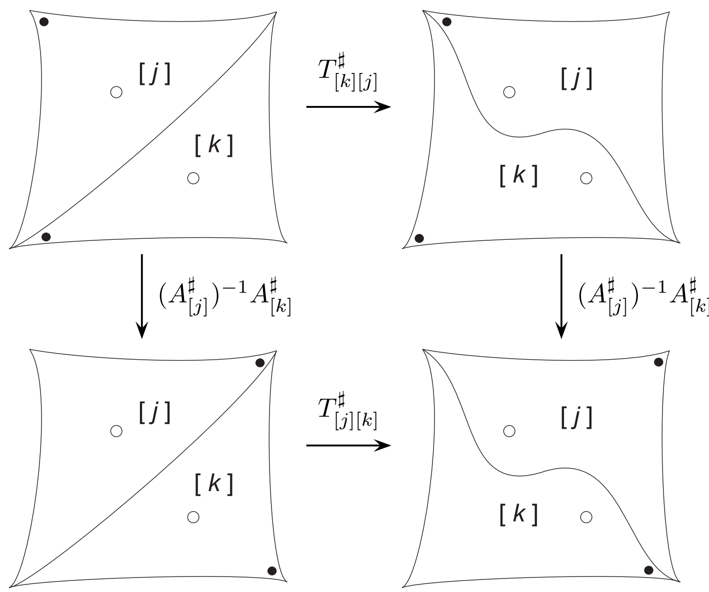


FIGURE 29. The relation $A^\sharp_{[j]}T^\sharp_{[j][k]}A^\sharp_{[k]} = A^\sharp_{[k]}T^\sharp_{[k][j]}A^\sharp_{[j]}$ on $\mathcal{F}tess^\sharp_{dot}$, here shown as $(A^\sharp_{[j]})^{-1}A^\sharp_{[k]}T^\sharp_{[k][j]} = T^\sharp_{[j][k]}(A^\sharp_{[j]})^{-1}A^\sharp_{[k]}$

of G_{dot} by $\mathbb{Z}$:

$$\begin{tikzcd} 1 \arrow[r] & B_\infty \arrow[r] \arrow[d] & G^\sharp_{dot} \arrow[r] \arrow[d] & G_{dot} \arrow[r] \arrow[d, "id"] & 1, \\ 1 \arrow[r] & \mathbb{Z} \arrow[r] & (G^\sharp_{dot})_{ab} \arrow[r] & G_{dot} \arrow[r] & 1. \end{tikzcd} \tag{5.33}$$

We denote the images of the projection map $G^\sharp_{dot} \to (G^\sharp_{dot})_{ab}$ by

$$G^\sharp_{dot} \to (G^\sharp_{dot})_{ab} : A^\sharp_{[j]} \mapsto \widetilde{A}^\sharp_{[j]}, \quad T^\sharp_{[j][k]} \mapsto \widetilde{T}^\sharp_{[j][k]}, \quad P_{(jk)} \mapsto \widetilde{P}^\sharp_{(jk)}, \quad \sigma_{[j][k]} \mapsto z, \tag{5.34}$$

where z is the generator of the center of $(G^\sharp_{dot})_{ab}$ which is isomorphic to $\mathbb{Z}$. It is now easy to obtain the following presentation of $(G^\sharp_{dot})_{ab}$, from Prop. 5.33, Def. 5.34, eq. (5.34), and the relations defining G_{dot} (Thm. 2.29 and Def. 2.30):

Proposition 5.37. *The group $(G^\sharp_{dot})_{ab}$ can be presented with the generators $\widetilde{A}^\sharp_{[j]}$, $\widetilde{T}^\sharp_{[j][k]}$, $\widetilde{P}^\sharp_{(jk)}$ (for $j,k \in \mathbb{Q}^\times$ with $j \neq k$) and z, with the following relations. First,*

$$(\widetilde{A}^\sharp_{[j]})^3 = id, \tag{5.35}$$

$$\widetilde{T}^\sharp_{[k][\ell]}\widetilde{T}^\sharp_{[j][k]} = \widetilde{T}^\sharp_{[j][k]}\widetilde{T}^\sharp_{[j][\ell]}\widetilde{T}^\sharp_{[k][\ell]}, \tag{5.36}$$

$$\widetilde{A}^\sharp_{[j]}\widetilde{T}^\sharp_{[j][k]}\widetilde{A}^\sharp_{[k]} = \widetilde{A}^\sharp_{[k]}\widetilde{T}^\sharp_{[k][j]}\widetilde{A}^\sharp_{[j]}, \tag{5.37}$$

$$\widetilde{T}^\sharp_{[j][k]}\widetilde{A}^\sharp_{[j]}\widetilde{T}^\sharp_{[k][j]} = z^{-1}\widetilde{A}^\sharp_{[j]}\widetilde{A}^\sharp_{[k]}\widetilde{P}^\sharp_{(jk)}, \tag{5.38}$$

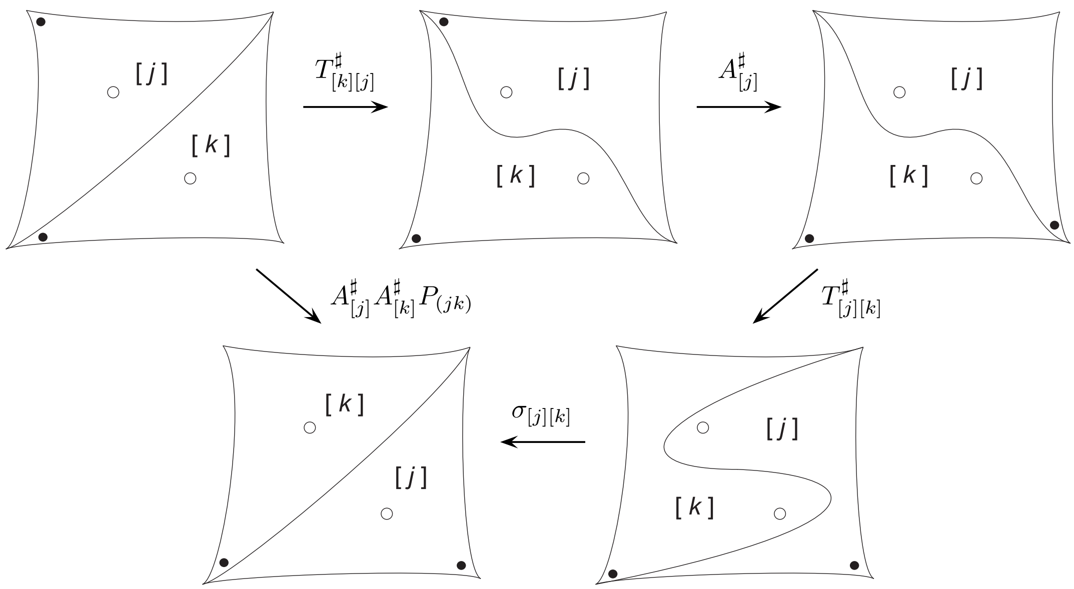


FIGURE 30. The relation $T^\sharp_{[j][k]} A^\sharp_{[j]} T^\sharp_{[k][j]} = (\sigma_{[j][k]})^{-1} A^\sharp_{[j]} A^\sharp_{[k]} P_{(jk)}$ on $\mathcal{F}tess^\sharp_{dot}$

where $j, k, \ell \in \mathbb{Q}^\times$ are mutually distinct. Also, the trivial relations for the index permutations $P^\sharp_{(jk)}$ hold as in (2.15)*, and any two words in the elementary moves whose collections of subscripts don' intersect with each other commute. Finally, we have the commuting relations*

$$[z, \widetilde{A}^\sharp_{[j]}] = [z, \widetilde{T}^\sharp_{[j][k]}] = [z, \widetilde{P}^\sharp_{(jk)}] = 1. \tag{5.39}$$

5.5. **The natural map from $(G^\sharp_{mark})_{ab}$ to $(G^\sharp_{dot})_{ab}$.** We will mimick the construction of the map $\mathbf{F} : G_{mark} \to G_{dot}$ (as done in §2.3), to obtain a map from $G^\sharp_{mark}$ to $G^\sharp_{dot}$ first, and then apply the relative abelianization. First, recall from Def. 5.23 and Prop. 5.24 that we have the following natural injective map

$$F^\sharp : \mathcal{F}tess^\sharp_{mark} \to \mathcal{F}tess^\sharp_{dot}; \tag{5.40}$$

just assign the dots and the triangle labels just as described in Def. 2.11. Then we now seek for a group homomorphism

$$\mathbf{F}^\sharp : G^\sharp_{mark} \to G^\sharp_{dot} \tag{5.41}$$

making the following diagram to commute for any $g \in G^\sharp_{mark}$:

$$\begin{tikzcd} \mathcal{F}tess^\sharp_{mark} \arrow[r, "F^\sharp"] \arrow[d, "g"] & \mathcal{F}tess^\sharp_{dot} \arrow[d, "\mathbf{F}^\sharp g"] \\ \mathcal{F}tess^\sharp_{mark} \arrow[r, "F^\sharp"] & \mathcal{F}tess^\sharp_{dot}, \end{tikzcd} \tag{5.42}$$

that is,

$$F^\sharp(g.\tau^\sharp_{mark}) = (\mathbf{F}^\sharp g).(F^\sharp(\tau^\sharp_{mark})), \quad \forall \tau^\sharp_{mark} \in \mathcal{F}tess^\sharp_{mark}, \quad \forall g \in G^\sharp_{mark}. \tag{5.43}$$

For any chosen $\tau^\sharp_{mark} \in \mathcal{F}tess^\sharp_{mark}$, the morphism $[F^\sharp(\tau^\sharp_{mark}), F^\sharp(g.\tau^\sharp_{mark})]$ between the two dotted $\sharp$-punctured tessellations can be broken into the composition of morphisms represented by elementary moves (Def. 5.32), and therefore leads to an element of $G^\sharp_{dot}$. However, it is a priori not clear that different choice of $\tau^\sharp_{mark} \in \mathcal{F}tess^\sharp_{mark}$ leads to the same element in $G^\sharp_{dot}$. By a similar argument as we used in §2.3, this is indeed true. By considering the actions of $\underline{\alpha}^\sharp, \underline{\beta}^\sharp$ and $\underline{\sigma}$ on e.g. the 'standard' marked $\sharp$-punctured tessellation of $\mathbb{D}^\sharp$, we can easily obtain the following (as an analog of (2.17)):

Proposition 5.38. *There exists a unique group homomorphism* $\mathbf{F}^\sharp$ (5.41) *satisfying* (5.43)*, and it is given by*

$$\mathbf{F}^\sharp : G^\sharp_{mark} \to G^\sharp_{dot} : \ \underline{\alpha}^\sharp \mapsto A^\sharp_{[-1]}(T^\sharp_{[-1][1]})^{-1} A^\sharp_{[1]} P^\sharp_{\gamma_\alpha}, \quad \underline{\beta}^\sharp \mapsto A^\sharp_{[-1]} P^\sharp_{\gamma_\beta}, \quad \underline{\sigma} \mapsto \sigma_{[-1][1]}. \tag{5.44}$$

Remark 5.39. *One can also deduce a characterization of the image of* $G^\sharp_{mark}$ *under* $\mathbf{F}^\sharp$ *inside* $G^\sharp_{dot}$*, similarly as in Prop. 2.37.*

Recall that $G^\sharp_{mark}$ and $G^\sharp_{dot}$ are extensions of G_{mark} and G_{dot} by the infinite braid group B_∞ (see Def. 5.10, Def. 5.11 and Prop. 5.12 for B_∞, which can easily be translated to the tessellation model). By abelianizing B_∞ we obtain the relative abelianizations $(G^\sharp_{mark})_{ab}$ and $(G^\sharp_{dot})_{ab}$, which are central extensions of G_{mark} and G_{dot} respectively. We now want to 'apply' this relative abelianization to the map (5.44) and obtain a map from $(G^\sharp_{mark})_{ab}$ to $(G^\sharp_{dot})_{ab}$:

Proposition 5.40. *There exists a unique group homomorphism* $\mathbf{F}^\sharp_{ab} : (G^\sharp_{mark})_{ab} \to (G^\sharp_{dot})_{ab}$ *making the following diagram to commute:*

$$\begin{tikzcd} G^\sharp_{mark} \arrow[r, "\mathbf{F}^\sharp"] \arrow[d] & G^\sharp_{dot} \arrow[d] \\ (G^\sharp_{mark})_{ab} \arrow[r, dotted, "\mathbf{F}^\sharp_{ab}"] & (G^\sharp_{dot})_{ab} \end{tikzcd} \tag{5.45}$$

where the vertical arrows are the relative abelianization homomorphisms ((5.22) *and* (5.33)*, whose images are given in Cor. 5.31 and* (5.34)*). It is given by*

$$\mathbf{F}^\sharp_{ab} : (G^\sharp_{mark})_{ab} \to (G^\sharp_{dot})_{ab} : \quad \underline{\widetilde{\alpha}}^\sharp \mapsto \widetilde{A}^\sharp_{[-1]}(\widetilde{T}^\sharp_{[-1][1]})^{-1} \widetilde{A}^\sharp_{[1]} \widetilde{P}^\sharp_{\gamma_\alpha}, \qquad \underline{\widetilde{\beta}}^\sharp \mapsto \widetilde{A}^\sharp_{[-1]} \widetilde{P}^\sharp_{\gamma_\beta}, \tag{5.46}$$

and furthermore, it is injective.

Proof. We first observe that B_∞ embeds as subgroups of both $G^\sharp_{mark}$ and $G^\sharp_{dot}$ (both these subgroups will be denoted by B_∞ here, by abuse of notation), and that the restriction of (5.41) $\mathbf{F}^\sharp : G^\sharp_{mark} \to G^\sharp_{dot}$ to these subgroups is the identity (each braiding is a homeomorphism class, hence induces the same elements of $G^\sharp_{mark}$ and $G^\sharp_{dot}$). The composition of $\mathbf{F}^\sharp$ and the relative abelianization $G^\sharp_{dot} \to (G^\sharp_{dot})_{ab} = G^\sharp_{dot}/[B_\infty, G^\sharp_{dot}]$ yields a map $\mathbf{F}^\sharp_0 : G^\sharp_{mark} \to (G^\sharp_{dot})_{ab}$. Since $[B_\infty, G^\sharp_{mark}]$ is in the kernel of $\mathbf{F}^\sharp_0$, the map $\mathbf{F}^\sharp_0$ factors through the relative abelianization

$G^\sharp_{mark} \to (G^\sharp_{mark})_{ab} = G^\sharp_{mark}/[B_\infty, G^\sharp_{mark}]$

$$\begin{tikzcd} G^\sharp_{mark} \arrow[r, "\mathbf{F}^\sharp"] \arrow[d] \arrow[dr, "\mathbf{F}^\sharp_0"] & G^\sharp_{dot} \arrow[d] \\ (G^\sharp_{mark})_{ab} \arrow[r, dotted, "\mathbf{F}^\sharp_{ab}"] & (G^\sharp_{dot})_{ab} \end{tikzcd} \tag{5.47}$$

hence yielding the unique group homomorphism (5.46) $\mathbf{F}^\sharp_{ab} : (G^\sharp_{mark})_{ab} \to (G^\sharp_{dot})_{ab}$. The formula (5.46) for $\mathbf{F}^\sharp_{ab}$ comes from the formula of $\mathbf{F}^\sharp$ (5.44) and the formulas for the two relative abelianization homomorphisms as in Cor. 5.31 and (5.34).

Suppose $x \in \ker \mathbf{F}^\sharp_{ab}$. Choose any of its lift X in $G^\sharp_{mark}$. Then $\mathbf{F}^\sharp(X) \in G^\sharp_{dot}$ projects to $1 \in (G^\sharp_{dot})_{ab}$ by the commutativity of the diagram (5.45), hence $\mathbf{F}^\sharp(X) \in [B_\infty, G^\sharp_{dot}]$. By the earlier observation about B_∞, we thus have $X \in [B_\infty, G^\sharp_{mark}]$, and therefore its projection x in $(G^\sharp_{mark})_{ab}$ is the identity element. Hence $\mathbf{F}^\sharp_{ab}$ is injective. □

The key point in the proof of Prop. 5.40 is the relationship between the subgroups (of braidings) of $G^\sharp_{mark}$ and $G^\sharp_{dot}$ isomorphic to B_∞; they come from the 'same' topological objects, and are just recorded in different ways.

5.6. **Identification of $\widehat{G}^{Kash}_{mark}$ with $(G^\sharp_{mark})_{ab} \cong$ '$T^\sharp_{ab}$.** Our goal is to construct the isomorphism between $\widehat{G}^{Kash}_{mark}$ and $T^\sharp_{ab}$, i.e. (5.12). Recall the construction procedure of the central extension of a group G from an almost G-homomorphism (see Def. 4.1), as shown in §4.1.

The group of our main interest here is (the Ptolemy-Thompson group) $G_{mark} \cong F_{mark}/R_{mark}$, where F_{mark} is the free group generated by α, β and R_{mark} is the normal subgroup generated by the relations for G_{mark} (see (2.9)). Then, using Kashaev's almost G_{mark}-homomorphism $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$ as given in Cor. 3.12 (in the sense of Def. 4.1), via the above mentioned procedure we obtained the central extension $\widehat{G}^{Kash}_{mark}$ of G_{mark}.

On the other hand, we have the central extension $(G^\sharp_{mark})_{ab}$ of G_{mark}, obtained in §5.3 as the relative abelianization of the extension $G^\sharp_{mark}$ of G_{mark} by the infinite braid group, where the braids come from the newly introduced $\sharp$-punctures. Since $G^\sharp_{mark}$ and $T^\sharp_{ab}$ are isomorphic via the dualizing relationship between the tessellation model and the ribbon tree model (Cor. 5.31), it suffices to prove $\widehat{G}^{Kash}_{mark} \cong (G^\sharp_{mark})_{ab}$ in order to prove the sought-for isomorphism $\widehat{G}^{Kash}_{mark} \cong T^\sharp_{ab}$ (5.12).

For the central extension $(G^\sharp_{mark})_{ab}$ of G_{mark}, we use the following tautological almost G_{mark}-homomorphism (in the sense of Def. 4.2)

$$F_{mark} \to (G^\sharp_{mark})_{ab} : \alpha \mapsto \underline{\widetilde{\alpha}}^\sharp, \quad \beta \mapsto \underline{\widetilde{\beta}}^\sharp, \tag{5.48}$$

obtained from a section $G_{mark} \to (G^\sharp_{mark})_{ab}$ which we can construct from (5.25). By Prop. 4.3, the tautological almost G_{mark}-homomorphism (5.48) yields the central extension $(G^\sharp_{mark})_{ab}$ by the procedure in §4.1. Since equivalent almost G_{mark}-homomorphisms yield isomorphic central extensions of G_{mark} (Prop. 4.5), it suffices to prove that the the almost G_{mark}-homomorphisms $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$ and (5.48) $F_{mark} \to (G^\sharp_{mark})_{ab}$ are equivalent.

This will be done in two steps, and what plays the role of a bridge between the two central extensions of G_{mark} is the central extension $(G^\sharp_{dot})_{ab}$ of the Kashaev group G_{dot} studied in §5.4,

which is the relative abelianzation of the extension $G^\sharp_{dot}$ of G_{dot} obtained by introducing the $\sharp$-punctures. Again, we use the following tautological G_{dot}-homomorphism

$$F_{dot} \to (G^\sharp_{dot})_{ab} : A_{[j]} \mapsto \widetilde{A}^\sharp_{[j]}, \quad T_{[j]} \mapsto \widetilde{T}^\sharp_{[j][k]}, \quad P_{(jk)} \mapsto \widetilde{P}^\sharp_{(jk)}, \tag{5.49}$$

where $G_{dot} = F_{dot}/R_{dot}$ and F_{dot} is the free group generated by $A_{[j]}$, $T_{[j][k]}$, $P_{(jk)}$ for $j,k \in \mathbb{Q}^\times$ ($j \neq k$) and R_{dot} is its normal subgroup generated by the relations in Thm. 2.29, obtained from a section $G_{dot} \to (G^\sharp_{dot})_{ab}$ coming from (5.31), (5.33) and (5.34). This tautological G_{dot}-homomorphism (5.49) yields the central extension $(G^\sharp_{dot})_{ab}$ of G_{dot} by the procedure in §4.1 (Prop. 4.3).

The two-step strategy can be sketched as

$$((5.48) : F_{mark} \to (G^\sharp_{mark})_{ab}) \sim ((5.49) : F_{dot} \to (G^\sharp_{dot})_{ab}) \sim ((3.17)\rho : F_{dot} \to GL(\mathscr{M})). \tag{5.50}$$

(See Thm. 3.10 for ρ.) The first $\sim$ in (5.50) will 'hold' roughly because $G^\sharp_{mark}$ and $G^\sharp_{dot}$ use the same topological setting (i.e. the $\sharp$-punctures) and are just written in different ways (which we studied in §5.5), and the second $\sim$ is by inspection of the presentation of $(G^\sharp_{dot})_{ab}$ and the relations of the operators for the projective representation (i.e. the images of the generators under $\rho : F_{dot} \to GL(\mathscr{M})$). To be more precise, the latter two almost G_{dot}-homomorphisms in (5.50) should be pre-composed with $F_{mark} \to F_{dot}$, and so the two equivalences of almost G_{mark}-homomorphisms that we shall prove are:

$$(F_{mark} \to (G^\sharp_{mark})_{ab}) \quad \overset{①}{\simeq} \quad (F_{mark} \to (G^\sharp_{dot})_{ab}) \quad \overset{②}{\simeq} \quad (\rho^{Kash} : F_{mark} \to GL(\mathscr{M})) \tag{5.51}$$

(see Cor. 3.12 for ρ^{Kash}). The map $F_{mark} \to F_{dot}$ that we pre-composed above is the following injective group homomorphism

$$F_{mark} \to F_{dot} : \alpha \mapsto A_{[-1]} T^{-1}_{[-1][1]} A_{[1]} P_{\gamma_\alpha}, \quad \beta \mapsto A_{[-1]} P_{\gamma_\beta} \tag{5.52}$$

coming from the formula (2.26) of the injective group homomorphism $\mathbf{F} : G_{mark} \to G_{dot}$ obtained in Prop. 2.35 (so (5.52) induces $\mathbf{F} : G_{mark} \to G_{dot}$). Therefore, the $F_{mark} \to (G^\sharp_{dot})_{ab}$ appearing in the middle of (5.51) is given by

$$F_{mark} \to (G^\sharp_{dot})_{ab} : \alpha \mapsto \widetilde{A}^\sharp_{[-1]} (\widetilde{T}^\sharp_{[-1][1]})^{-1} \widetilde{A}^\sharp_{[1]} \widetilde{P}^\sharp_{\gamma_\alpha}, \quad \beta \mapsto \widetilde{A}^\sharp_{[-1]} \widetilde{P}^\sharp_{\gamma_\beta}. \tag{5.53}$$

For completeness, we record the third almost G_{mark}-homomorphism appearing in (5.51), i.e. $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$, is given by

$$\rho^{Kash} : F_{mark} \to GL(\mathscr{M}) : \alpha \mapsto \rho^{Kash}(\alpha) = \widehat{\alpha}, \quad \beta \mapsto \rho^{Kash}(\beta) = \widehat{\beta}; \tag{5.54}$$

see Def. 4.10 for the notations $\widehat{\alpha}, \widehat{\beta}$. Meanwhile, the latter two maps $F_{mark} \to (G^\sharp_{dot})_{ab}$ and $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$ of (5.51) are indeed almost G_{mark}-homomorphisms by the first part of Lemma 4.6, because they are obtained by pre-composing the (latter two) almost G_{dot}-homomorphisms in (5.50) with the group homomorphism $F_{mark} \to F_{dot}$ (5.52), which satisfies the condition of Lemma 4.6 because (5.52) takes R_{mark} to R_{dot} (since it induces $G_{mark} \to G_{dot}$).

The key point in this subsection is the proof of the equivalences ① and ② of (5.51). We first prove the first equivalence ① of (5.51), using our knowledge about the relationship between $(G^\sharp_{mark})_{ab}$ and $(G^\sharp_{dot})_{ab}$ as studied in §5.5.

Proposition 5.41. *We have the following equivalence of the almost G_{mark}-homomorphisms*

$$((5.48) : F_{mark} \to (G^\sharp_{mark})_{ab}) \simeq ((5.53) : F_{mark} \to (G^\sharp_{dot})_{ab}), \tag{5.55}$$

via the injective group homomorphism $\mathbf{F}^\sharp_{ab} : (G^\sharp_{mark})_{ab} \to (G^\sharp_{dot})_{ab}$ (5.46)*. (see Def. 4.4 for the 'equivalence'.)*

Proof. By looking at the formulas (5.52), (5.48), (5.49), and (5.46), and since (5.53) was defined to be the composition of (5.52) and (5.49), we can see that the following diagram commutes:

$$\begin{tikzcd} F_{mark} \arrow[r, "(5.52)"] \arrow[d, "(5.48)"'] \arrow[dr, dotted, "(5.53)"] & F_{dot} \arrow[d, "(5.49)"] \\ (G^\sharp_{mark})_{ab} \arrow[r, hook, "\mathbf{F}^\sharp_{ab}"'] & (G^\sharp_{dot})_{ab} \end{tikzcd} \tag{5.56}$$

Since the bottom map $\mathbf{F}^\sharp_{ab}$ (5.46) is injective (Prop. 5.40), by Lemma 4.6 the almost G_{mark}-homomorphism (5.48) $F_{mark} \to (G^\sharp_{mark})_{ab}$ is equivalent to its post-composition with $\mathbf{F}^\sharp_{ab}$, namely (5.53) $F_{mark} \to (G^\sharp_{dot})_{ab}$, via $\mathbf{F}^\sharp_{ab}$. □

As mentioned earlier, the second equivalence ② of (5.51) is just by inspection of the relations of the Kashaev projective operators and the presentation of the 'geometric Kasahev group $(G^\sharp_{dot})_{ab}$:

Proposition 5.42. *We have the following equivalence of the almost G_{mark}-homomorphisms*

$$((5.53) : F_{mark} \to (G^\sharp_{dot})_{ab}) \simeq (\rho^{Kash} : F_{mark} \to GL(\mathscr{M})), \tag{5.57}$$

via the group homomorphism

$$(G^\sharp_{dot})_{ab} \to GL(\mathscr{M}) : \widetilde{A}^\sharp_{[j]} \mapsto \rho(A_{[j]}),\ \widetilde{T}^\sharp_{[j][k]} \mapsto \rho(T_{[j][k]}),\ \widetilde{P}^\sharp_{(jk)} \mapsto \rho(P_{(jk)}),\ z \mapsto \zeta^{-1}. \tag{5.58}$$

(see Def. 4.4 for the 'equivalence'.)

Proof. By inspection of the equations appearing in Propositions 3.11 and 5.37, the tautological almost G_{dot}-homomorphism $F_{dot} \to (G^\sharp_{dot})_{ab}$ (5.49) is equivalent to the following almost G_{dot}-homomorphism (coming from Kashaev's almost linear representation of G_{dot})

$$\rho : F_{dot} \to GL(\mathscr{M})$$

in (3.17), as defined in Thm. 3.10, i.e. we have the equivalence

$$((5.49) : F_{dot} \to (G^\sharp_{dot})_{ab}) \simeq (\rho : F_{dot} \to GL(\mathscr{M})) \tag{5.59}$$

(see Def. 4.4 for the 'equivalence') via the group homomorphism $(G^\sharp_{dot})_{ab} \to GL(\mathscr{M})$ (5.58). By pre-composing this equivalence (5.59) with the group homomorphism $F_{mark} \to F_{dot}$ (5.52), we get the desired result (5.57) (see Lem. 4.6 for pre-composition of equivalent almost group homomorphisms and a group homomorphism; we already saw before that (5.52) satisfies the condition of the lemma). □

Since the equivalence of almost group homomorphisms is an equivalence relation (Prop. 4.5), from Propositions 5.41 and 5.42 (i.e. ① and ② of (5.51) we get:

Corollary 5.43. *We have the following equivalence of the almost G_{mark}-homomorphisms*

$$((5.48) : F_{mark} \to (G^\sharp_{mark})_{ab}) \simeq (\rho^{Kash} : F_{mark} \to GL(\mathscr{M})), \tag{5.60}$$

(see (5.54) *for ρ^{Kash}) via the group homomorphism*

$$(G^\sharp_{mark})_{ab} \to GL(\mathscr{M}) : \quad \underline{\widetilde{\alpha}}^\sharp \mapsto \widehat{\alpha}, \qquad \underline{\widetilde{\beta}}^\sharp \mapsto \widehat{\beta} \tag{5.61}$$

(obtained as the composition of (5.46) *and* (5.58)*; see Def. 4.10 for the definition of $\widehat{\alpha}$, $\widehat{\beta}$).*

Cor. 5.43 gives the equivalence of the two almost G_{mark}-homomorphisms, $F_{mark} \to (G^\sharp_{mark})_{ab}$ (5.48) and $\rho^{Kash} : F_{mark} \to GL(\mathscr{M})$ (5.54). The first one is the tautological G_{mark}-homomorphism, hence yields by the procedure in §4.1 the central extension $(G^\sharp_{mark})_{ab}$ of G_{mark} (see Prop. 4.3). The second one yields the central extension $\widehat{G}^{Kash}_{mark}$ of G_{mark}; see Thm. 4.9.

Meanwhile, by Prop. 4.5, any two equivalent G_{mark}-homomorphisms yield isomorphic central extensions of G_{mark}. Therefore these two central extensions $(G^\sharp_{mark})_{ab}$ and $\widehat{G}^{Kash}_{mark}$ are isomorphic, and we also have an explicit isomorphism between them as mentioned in Prop. 4.5, based on the group homomorphism (5.61). Thus we finally obtain the following identification.

Theorem 5.44. *Let the group $\widehat{G}^{Kash}_{mark}$ be presented with the generators $\overline{\alpha}$, $\overline{\beta}$, z and relations in* (4.43)*. Then we have the following isomorphism of the two central extensions of G_{mark}:*

$$(G^\sharp_{mark})_{ab} \to \widehat{G}^{Kash}_{mark} : \quad \underline{\widetilde{\alpha}}^\sharp \mapsto \overline{\alpha}, \qquad \underline{\widetilde{\beta}}^\sharp \mapsto \overline{\beta}, \tag{5.62}$$

where $\underline{\widetilde{\alpha}}^\sharp$ and $\underline{\widetilde{\beta}}^\sharp$ are as defined in Cor. 5.31.

Proof. One can see that $\underline{\widetilde{\alpha}}^\sharp$ and $\underline{\widetilde{\beta}}^\sharp$ are the lifts of α and β of G_{mark} to the central extension $(G^\sharp_{mark})_{ab}$ and that $\overline{\alpha}$ and $\overline{\beta}$ are the lifts of α and β of G_{mark} to the central extension $\widehat{G}^{Kash}_{mark}$, via the lifting maps described in (4.5) (in fact, one can recall from (4.43) that $\overline{\alpha}$ and $\overline{\beta}$ are defined precisely as the lifts in the finitely presented group $\widehat{G}^{Kash}_{mark}$ of α and β corresponding to the almost G_{mark}-homomorphism $\rho^{Kash} : \alpha \mapsto \widehat{\alpha}$, $\beta \mapsto \widehat{\beta}$ (see Def. 4.10)). $\square$

Corollary 5.45. *Using the identification $(G^\sharp_{mark})_{ab} \to T^\sharp_{ab}$* (5.24)*, we get the following isomorphism*

$$\widehat{G}^{Kash}_{mark} \to T^\sharp_{ab} : \quad \overline{\alpha} \mapsto \widetilde{\alpha}^\sharp, \qquad \overline{\beta} \mapsto \widetilde{\beta}^\sharp, \tag{5.63}$$

(see Notation 5.16) thus finally proving the sought-for isomorphism (5.12).

In our above proof for the isomorphism (5.63) (hence (5.12)), we didn't check all the relations of α, β for the group G_{mark} (in (2.9)), unlike our algebraic proof in §4.2 of the main theorem $\widehat{G}^{Kash}_{mark} \cong T_{3,2,0,0}$ (4.13) of the present paper; the only actual computation involved is the computation of the four simple relations for the generators of $G^\sharp_{dot}$ (and therefore of $(G^\sharp_{dot})_{ab}$), which was quite easy to check by looking at the pictures (Prop. 5.33). So one may think that we get away with this simple graphical proof and that all the algebraic computations in §4.2 were unnecessary.

However, the identification (5.63) itself alone does not let us compute the extension class of $\widehat{G}^{Kash}_{mark}$, because it doesn't provide a presentation of it in terms of generators and relations. We further need to consult Funar-Kapoudjian's result $T^\sharp_{ab} \cong T_{3,2,0,0}$ in [FuKap2] (i.e. a presentation of $T^\sharp_{ab}$) in order to really identify the extension class of $\widehat{G}^{Kash}_{mark}$. And Funar-Kapoudjian's proof in [FuKap2] of $T^\sharp_{ab} \cong T_{3,2,0,0}$ involves checking all the α, β relations (for G_{mark} group), although their proof is graphical rather than algebraic. Therefore the computation in §4.2 can be thought of as an algebraic counterpart of Funar-Kapoudjian's graphical computation of the lifted relations of α, β, i.e. their graphical proof of $T^\sharp_{ab} \cong T_{3,2,0,0}$.

6. Further questions

In this section, we address some open questions and conjectures.

6.1. **The relationships $\rho^{CF} \leftrightarrow \rho^{Kash}$, $T^* \leftrightarrow T^\sharp$, and $B_{\infty-1} \leftrightarrow B_\infty$.** Funar and Kapoudjian [FuKap2] showed that $T^\sharp$ is related to T^* in the same way as the braid group B_n is related to B_{n-1}, for infinite n (for B_∞, see Def. 5.10 and the remarks following it, and Def. 5.11). We review §2.4 of [FuKap2]. Roughly speaking, one obtains T^* by considering the mapping classes of $T^\sharp$ associated to those homeomorphisms fixing one specific puncture of $D^\sharp$, and by viewing them as mapping classes of $D^\sharp$ union that puncture. Specifically, denote by $T^\sharp_*$ the subgroup of $T^\sharp$ formed by those homeomorphism classes that keep fixed the puncture in the interior of the hexagon (s) of $D^\sharp$ (or equivalently, puncture in the triangle [1] of $\mathbb{D}^\sharp$) (see §5.1 and §5.3). Let $B_{\infty,1} \subset B_\infty$ denote the subgroup of braids that keep fixed the puncture q, and $B_{\infty-1}$ denote the infinite braid on the punctures of $D^\sharp \cup \{q\}$. There is an obvious projection map $B_{\infty,1} \to B_{\infty-1}$ that consists in deleting the strand over q.

Proposition 6.1 ([FuKap2])**.** *We have a commutative diagram with exact lines and columns:*

(6.1)
$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & B_{\infty,1} \arrow[r] \arrow[d] & B_{\infty-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & T^\sharp_* \arrow[r] \arrow[d] & T^* \arrow[r] \arrow[d] & 1 \\
 & 1 \arrow[r] & T \arrow[r] \arrow[d] & T \arrow[r] \arrow[d] & 1 \\
 & & 1 & 1 &
\end{tikzcd}$$

where $\mathcal{F}$ is a free group, normally generated by $\sigma^2 = (\alpha^\sharp)^4$ as a subgroup of $T^\sharp_*$.

The central extensions $\widehat{G}^{CF}_{mark}$ and $\widehat{G}^{Kash}_{mark}$ of $G_{mark} \cong T$ induced respectively by the almost linear (projective) representations ρ^{CF} and ρ^{Kash} have extensions classes 12χ and 6χ respectively, in $H^2(G_{mark})$. It's not clear what exactly is responsible for this discrepancy. There may be some shift of the generator of the center of $\widehat{G}^{CF}_{mark}$ and the generator of the center of $\widehat{G}^{Kash}_{mark}$, for example one is the square of the other. Since $\widehat{G}^{CF}_{mark} \cong T^*_{ab}$ and $\widehat{G}^{Kash}_{mark} \cong T^\sharp_{ab}$, and from the above relationship between T^* and $T^\sharp$, maybe the difference can be explained using that between $B_{\infty-1}$ and B_∞ somehow. We record here what we have so far:

(6.2)
$$\begin{array}{r|l|l|l}
\rho^{CF} : F_{mark} \to GL(\mathscr{V}) & \mathscr{V} \equiv L^2(\mathbb{R}^{\tau^{(1)}}) & \widehat{T} \cong T^*_{ab} \text{ with } c_{\widehat{T}} = 12\chi & B_{\infty-1} = \text{braids of } \tau^{(1)} \\
\rho^{Kash} : F_{mark} \to GL(\mathscr{M}) & \mathscr{M} \equiv L^2(\mathbb{R}^{\tau^{(2)}}) & \widetilde{T} \cong T^\sharp_{ab} \text{ with } c_{\widehat{T}} = 6\chi & B_{\infty} = \text{braids of } \tau^{(2)}
\end{array}$$

where

$$\tau^{(1)} = \{\text{edges of } \tau\} = \mathbb{Q} \setminus \{0,1\}, \quad \tau^{(2)} = \{\text{triangles of } \tau\} = \mathbb{Q}^\times = \mathbb{Q} \setminus \{0\}. \tag{6.3}$$

6.2. **Finite type surfaces.** What is studied in the present paper and [FuS] is the Teichmüller theory and mapping class group on the 'infinite-type' surface, namely, either ribbon graph, or open unit disc with a certain restriction on the boundary behavior. The quantization of the universal Teichmüller spaces provides projective (more precisely, 'almost linear') representations

of the group $T \cong G_{mark}$, which is the 'asymptotically rigid' mapping class group of this infinite surface, which in turn leads to the central extensions of this group.

Now, there's a similar story for finite type surfaces, leading to central extensions of the relevant (genuine) mapping class groups. Let $\Gamma^s_{g,r}$ be the mapping class group of a surface $\Sigma^s_{g,r}$ of genus g with r boundary components and s punctures. These are mapping classes of homeomorphisms which fix the boundary pointwise and permute the punctures. For $(2g-2+2s) > 0$ (so that there exist complete hyperbolic metrics on $\Sigma^s_{g,r}$), the projective representations of $\Gamma^s_{g,r}$ are constructed by Kashaev [Kas1] [Kas2], which lead to the central extensions $\widetilde{\Gamma}^s_{g,r}$ via the process described in §4.1 (minimal reductions of the ones obtained as pullback by the canonical projection $GL(\mathcal{H}) \to PGL(\mathcal{H})$ where $\mathcal{H}$ is the relevant Hilbert space). This central extension is studied by Funar and Kashaev [FuKas], and their main result is the following theorem:

Theorem 6.2. *The the cohomology class of the central extension $\widetilde{\Gamma}^s_{g,r}$ of $\Gamma^s_{g,r}$ by a cyclic Abelian group $A \subset \mathbb{C}^*$ (generated by ζ^{-6}) is*

$$c_{\widetilde{\Gamma}^s_{g,r}} = 12\chi + \sum_{i=1}^{s} e_i \in H^2(\Gamma^s_{g,r}; A) \tag{6.4}$$

if $g \geq 2$ and $s \geq 4$. Here χ and e_i are one fourth of the Meyer signature class and respectively the i-th Euler class with A coefficients.

See [FuKas] and references therein (e.g. Korkmaz-Stipsicz [KoSt], Corollary 4.4) for the results about $H^2(\Gamma^s_{g,r})$. For example, for $g \geq 4$ we have $H^2(\Gamma^s_{g,r}) \cong \mathbb{Z}^{s+1}$, generated by the classes χ and e_i. For $g = 3$ we have $\mathbb{Z}^{s+1} \subset H^2(\Gamma^s_{g,r})$, while $H^2(\Gamma^s_{2,r})$ contains the subgroup $\mathbb{Z}/10\mathbb{Z} \oplus \mathbb{Z}^s$, whose torsion part is generated by χ and whose free part is generated by the Euler classes. The Universal Coefficients Theorem permits then to compute $H^2(G; A)$ for every Abelian group A.

The construction of Funar and Kashaev [FuKas] uses the Kashaev quantization of the Teichmüller spaces, thus involving the operators like $\mathbf{A}_\cdot$, $\mathbf{T}_{\cdot,\cdot}$, $\mathbf{P}_{\cdot\cdot}$, and their proof is similar to our algebraic proof in §4.2. Instead of the Ptolemy-Thompson group relations (2.9), they computed the usual relations for the mapping class groups, e.g. the chain relations and the lantern relations of the Dehn twists.

Naturally, one might consider using the Chekhov-Fock quantization (or the Fock-Goncharov formulation) of the Teichmüller space of a finite type surface, and compute the presentation of the central extension of the mapping class group $\Gamma^s_{g,r}$ thus induced, and therefore also the corresponding extension class. It is expected ([Fu]) to yield the same result as in the case of the Kashaev quantization; then it would remain to clarify why the two formulations yield distinct cohomology class only in the 'limiting' case.

Yet another interesting question raised by Funar [Fu] is the 'geometric' realization of these central extensions of the mapping class groups of finite type surfaces. In the case of the 'infinite type' surface studied in [FuS] and the present paper, the induced central extensions of the 'mapping class group' T are identified with the relative abelianizations of the extensions $T^*, T^\sharp$ of T by the infinite braid group, which are 'geometrically' realized by introducing certain punctures in two different ways. One can also introduce the punctures in an analogous way for finite type surface cases to construct the extensions of the mapping class groups by braid groups of finite number of strands, but in this case the relative abelianizations yield the central extensions of the mapping class groups by $\mathbb{Z}/2\mathbb{Z}$, not by $\mathbb{Z}$. This phenomenon only occurs for positive genus case; see [BeFu]. So one may introduce some extra punctures, e.g. using the two

types of punctures at the same time, to obtain a larger extension which would yield the desired result by the relative abelianization (suggested by [Fu]).

6.3. **Quantum Teichmüller space as the space of intertwiners of the quantum plane algebra.** Kashaev's operator $\rho(T_{[j][k]})$, which involves the quantum dilogarithm function, solves the so-called pentagon relation (see (3.31)). It is well known that the pentagon relation appears as the associativity constraint of a tensor category. So, one may ask if Kashaev's operator arises from a special tensor category. Recently Igor Frenkel and the author gave a positive answer to this question in [FrKi], and the description of the relevant tensor category is rather simple. It is the category of the 'integrable' (i.e. nicely bahaved) representations of the so-called 'modular double' (see [Fa]) of the quantum plane Holf algebra, where the quantum plane is the self-adjoint version of its unitary counterpart, the (more familiar) quantum torus algebra, which is the most basic noncommutative (Hopf) algebra. Let us first briefly review that result here.

The relevant algebra is denoted by $B_{q\widetilde{q}}$, generated by the four generators $X, Y, \widetilde{X}, \widetilde{Y}$, with the algebraic relations

$$XY = q^2 YX, \quad \widetilde{X}\widetilde{Y} = \widetilde{q}^2 \widetilde{Y}\widetilde{X}, \tag{6.5}$$

$$[X, \widetilde{X}] = [X, \widetilde{Y}] = [Y, \widetilde{X}] = [Y, \widetilde{Y}] = 1, \tag{6.6}$$

where

$$q = e^{\pi i b^2}, \quad \widetilde{q} = e^{\pi i b^{-2}}, \quad b \in (0, \infty),\, b^2 \notin \mathbb{Q}. \tag{6.7}$$

We also impose the operator-theoretic relations

$$X^{1/b^2} = \widetilde{X}, \quad Y^{1/b^2} = \widetilde{Y}, \quad X^* = X, \quad Y^* = Y. \tag{6.8}$$

So $B_{q\widetilde{q}}$ can be thought of as a positive Borel subalgebra of $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, the modular double of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$. Then $B_{q\widetilde{q}}$ has a unique irreducible 'integrable' representation on

$$\mathscr{H} \equiv L^2(\mathbb{R}, dx) \tag{6.9}$$

via the following operators:

$$\pi(X) = e^{-2\pi b p}, \quad \pi(Y) = e^{2\pi b x}, \quad \pi(\widetilde{X}) = e^{-2\pi b^{-1} p}, \quad \pi(\widetilde{Y}) = e^{2\pi b^{-1} x}, \tag{6.10}$$

where $p = \frac{1}{2\pi i}\frac{d}{dx}$ as also used in the current paper with a different notation (hence p, x form the Heisenberg algebra $[p, x] = \frac{1}{2\pi i}$). One can think of the tensor product representations via the coproduct structure of $B_{q\widetilde{q}}$ since it is in fact a Hopf algebra:

$$\Delta X = X \otimes X, \quad \Delta Y = Y \otimes X + 1 \otimes Y, \quad \Delta\widetilde{X} = \widetilde{X} \otimes \widetilde{X}, \quad \Delta\widetilde{Y} = \widetilde{Y} \otimes \widetilde{X} + 1 \otimes \widetilde{Y}. \tag{6.11}$$

When we take the tensor product representation $\mathscr{H} \otimes \mathscr{H}$, we expect that it decomposes to irreducibles, i.e. copies of $\mathscr{H}$. Instead of the direct sum, this has to be the 'direct integral' (and therefore the category we're dealing with is not a traditionally studied tensor category; it is a 'continuous' version of a tensor category), and it is realized as an intertwining map

$$\mathscr{H} \otimes \mathscr{H} \xrightarrow{\sim} M \otimes \mathscr{H}, \tag{6.12}$$

where

$$M \cong Hom_{B_{q\widetilde{q}}}(\mathscr{H}, \mathscr{H} \otimes \mathscr{H}) \tag{6.13}$$

is a 'multiplicity module', which is a trivial $B_{q\widetilde{q}}$-module also realized as $L^2(\mathbb{R})$. The map (6.12) is realized in [FrKi] as a certain (unitary) integral transformation on $L^2(\mathbb{R}^2)$, whose integral kernel involves the quantum dilogarithm function. Then the canonical isomorphism

$$(\mathscr{H}_1 \otimes \mathscr{H}_2) \otimes \mathscr{H}_3 \cong \mathscr{H}_1 \otimes (\mathscr{H}_2 \otimes \mathscr{H}_3) \tag{6.14}$$

yields the operator

$$\mathbf{T} : M^6_{43} \otimes M^4_{12} \xrightarrow{\sim} M^5_{23} \otimes M^6_{15}, \tag{6.15}$$

where the indices for M indicates

$$M^\ell_{jk} \cong Hom_{B_{q\tilde{q}}}(\mathscr{H}_\ell, \mathscr{H}_j \otimes \mathscr{H}_k). \tag{6.16}$$

This operator $\mathbf{T}$ on $L^2(\mathbb{R}^2)$ satisfies the pentagon relation, by construction. One of Frenkel-Kim's results is that this $\mathbf{T}$ is essentially equivalent to Kashaev's operator, i.e. there is a simple unitary transformation U given by an explicit formula (some analog of the Fourier transformation) such that

$$U^{-1}\,\mathbf{T}_{[j][k]}\,U = \rho(T_{[j][k]}). \tag{6.17}$$

Moreover, this category of representations of $B_{q\tilde{q}}$ is 'rigid', meaning that there is a left and right duals $\mathcal{H}'$ and ${}'\mathcal{H}$ satisfying certain properties. They are also realized as $L^2(\mathbb{R})$, and isomorphic to $\mathcal{H}$ via non-unitary maps. Using these isomorphisms, Frenkel-Kim constructed the operator

$$\mathbf{A} : M^3_{12} \cong Hom_{B_{q\tilde{q}}}(\mathscr{H}_3, \mathscr{H}_1 \otimes \mathscr{H}_2) \xrightarrow{\sim} Hom_{B_{q\tilde{q}}}(\mathscr{H}_1, \mathscr{H}_2 \otimes \mathscr{H}_3) \cong M^1_{23}, \tag{6.18}$$

which is realized as a certain (non-unitary) integral transformation on $L^2(\mathbb{R})$. Then they proved the identities

$$\mathbf{A}^3 = id, \quad \mathbf{A}_1\mathbf{T}_{12}\mathbf{A}_2 = \mathbf{A}_2\mathbf{T}_{21}\mathbf{A}_1, \quad \mathbf{T}_{12}\mathbf{A}_1\mathbf{T}_{21} = \mathbf{A}_1\mathbf{A}_2 P_{(12)}, \tag{6.19}$$

using the representation theory, mostly just diagram chasing. By the same unitary operator U as in (6.17), this $\mathbf{A}$ operator is almost equivalent to Kashaev's $\rho(A_{[j]})$ operator:

$$U^{-1}\,\mathbf{A}^{(0)}_{[j]}\,U = \rho(A_{[j]}), \tag{6.20}$$

where

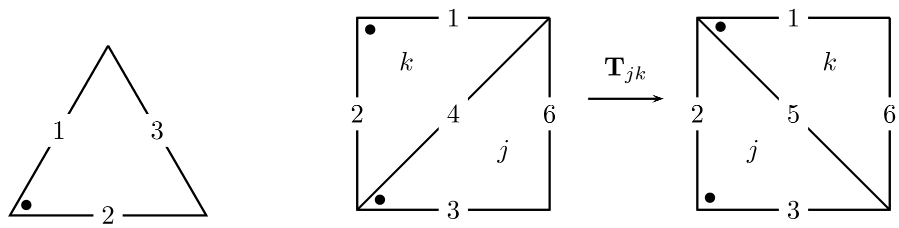
$$\mathbf{A}^{(m)}_{[j]} = (\zeta^{m^2-1}\mathbf{A}\,e^{\pi(b+b^{-1})(m-1)p}), \quad m \in \mathbb{R}, \tag{6.21}$$

is a family of operators (for which Frenkel-Kim obtained their result) which are related to $\mathbf{A}$ in a simple manner, and $\zeta = e^{-\pi i(b+b^{-1})^2/12}$ as in (3.34).

They also showed how to graphicaly encode their construction, in terms of the dotted triangulations (tessellations) of the surfaces. The space $M^3_{12} \cong Hom_{B_{q\tilde{q}}}(\mathscr{H}_3, \mathscr{H}_1 \otimes \mathscr{H}_2)$ is encoded as the triangle with sides labeled by $1, 2, 3$, with a dot $\bullet$ in the corner facing the edge labeled by 3; see Fig. 31A. Then it's easy to see that the $\mathbf{A}$ operator (6.18) is just moving the dot of this triangle counterclockwise to the next corner (so that the dot is now facing the edge labeled by 1). The $\mathbf{T}$ operator (6.15) involves a quadrilateral consisting of two triangles, and it acts as replacing the diagonal of the quadrilateral by the other diagonal, with the appropriate configuration of dots; see Fig. 31B. In particular, one can quickly grasp that these are precisely same as the elementary moves $A_{[j]}$ and $T_{[j][k]}$ (of G_{dot}) on $\mathcal{F}tess_{dot}$, as in Def. 2.27.

As mentioned in §6.2 we consider a surface of genus g with r boundary components and s punctures with distinguished points on the boundary ([P3]), and the quantization of the Teichmüller space of such a surface. We first consider a simple example of genus 0 with one boundary component and n distinguished points on the boundary; we view it as 'n-gon' (as in §6.2).

Let the sides (outer edges) of the n-gon be enumerated counterclockwise from 0 to $n-1$, where the i-th edge corresponds to the representation $\mathscr{H}_i \cong \mathscr{H}$. The special role of the 0-edge is accounted in the decoration of the n-gon with $n-2$ dots $\bullet$ near all the vertices except

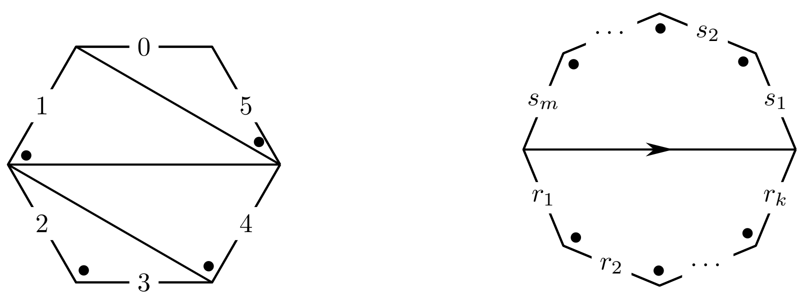


(A) The triangle for M^3_{12} (B) The move for $\mathbf{T} : M^6_{43} \otimes M^4_{12} \to M^5_{23} \otimes M^6_{15}$

FIGURE 31. The graphical presentation of the space of intertwiners

the two endpoints of the 0-edge. It is well known that various triangulations of n-gon are in one-to-one correspondence with various arrangements of the parentheses in the product $\mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_{n-1}$, where the role of the dots is as in Fig. 31A. Note that the choice of a triangulation also uniquely determines the placement of each dot near every vertex in a particular triangle of the triangulation. For example, the decorated triangulation of a 6-gon as in Fig. 32A corresponds to the following arrangement of the parentheses

$$Hom_{B_{q\tilde{q}}}(\mathscr{H}_0, (\mathscr{H}_1 \otimes ((\mathscr{H}_2 \otimes \mathscr{H}_3) \otimes \mathscr{H}_4)) \otimes \mathscr{H}_5). \tag{6.22}$$



(A) An example of a dotted triangulation of a 6-gon

(B) Quantization of an n-gon with a distinguished oriented diagonal

FIGURE 32. Quantization of n-gon's

By forgetting the parentheses, we are now able to identify the quantum Teichmüller space of the n-gon directly with the space of intertwining operators

$$Hom_{B_{q\tilde{q}}}(\mathscr{H}_0,\, \mathscr{H}_1 \otimes \mathscr{H}_2 \otimes \cdots \otimes \mathscr{H}_{n-1}). \tag{6.23}$$

We could also consider a dual construction of the quantum Teichmüller space of the n-gon identifying it with the space of intertwining operators

$$Hom_{B_{q\tilde{q}}}(\mathscr{H}_{n-1} \otimes \mathscr{H}_{n-2} \otimes \cdots \otimes \mathscr{H}_1,\, \mathscr{H}_0). \tag{6.24}$$

The isomorphism of the two quantizations results from the isomorphisms $\mathscr{H}_i \cong \mathscr{H}_i'$ for all i, and the isomorphisms of Hom's with their duals. Combining the two pictures together we obtain the quantization of the n-gon with a distinguished oriented diagonal rather than an edge. In this case the quantum Teichmüller space becomes

$$Hom_{B_{q\tilde{q}}}(\mathscr{H}_{s_m} \otimes \cdots \otimes \mathscr{H}_{s_1},\, \mathscr{H}_{r_1} \otimes \cdots \otimes \mathscr{H}_{r_k}), \tag{6.25}$$

where $k+m=n$; see Fig. 32B. When we consider the 'limit' of n-gon's as $n \to \infty$ (then Fig. 32B would 'go to' Fig. 2A), the corresponding 'limit' of the quantum Teichmüller spaces would be the quantum universal Teichmüller space, which we've been dealing with in the present paper. Thus, formally, the quantum universal Teichmüller space can be identified with

$$Hom_{B_{q\widetilde{q}}}\left(\bigotimes_{s \in -\mathbb{Q}_{\geq 0}^{-1}} \mathscr{H}_s, \bigotimes_{r \in \mathbb{Q}_{\geq 0}} \mathscr{H}_r \right), \tag{6.26}$$

where the product is taken in the increasing order of the indices in $\mathbb{Q}_{\geq 0}$ and in the decreasing order of the indices in $-\mathbb{Q}_{\geq 0}^{-1} \equiv \mathbb{Q}_{<0} \cup \{\infty\}$. By dualizing all $\mathscr{H}$'s in the first product of (6.26) we can obtain even more symmetric form of the quantum universal Teichmüller space, namely

$$Inv\left(\bigotimes_{r \in \mathbb{Q} \cup \{\infty\}} \mathscr{H}_r \right), \tag{6.27}$$

where the factors are ordered in the counterclockwise direction of the circle with the assumed cyclic symmetry, and Inv means the invariant subspace (=the subspace of vectors on which the Hopf algebra $B_{q\widetilde{q}}$ acts trivially, i.e. by the counit). For a discussion on how to make sense of the infinite tensor product appearing in (6.27), see [FrKi] §6.2.

Choosing a (marked) tessellation corresponds to a certain 'choice of parentheses' of the infinite product in (6.27), and $\mathbf{T}$ and $\mathbf{A}$ correspond to changes of the parenthesizing and the ordering of the factors, in some sense. Now, take α and β, which can be represented by $\mathbf{A}, \mathbf{T}, \mathbf{P}$ by the formula (2.26), and then we can try to interpret these in terms of the effect on the infinite product in (6.27), i.e. at the level of $\mathscr{H}$'s instead of M's (where $M = Hom_{B_{q\widetilde{q}}}(\mathscr{H}, \mathscr{H} \otimes \mathscr{H})$), thus adding one more line to (6.2) in an appropriate sense. This will provide a representation-theoretic interpretation of the Ptolemy-Thompson group elements. This problem is suggested to the author by Frenkel during his course [Fr], and the present paper grew out of the trials on answering this question. One can also try this for the finite-type surfaces, when the tensor product only involves finite number of factors.

As suggested by Frenkel [Fr] and Funar [Fu], a very interesting and probably difficult problem is the construction of representations of T^* and $T^\sharp$, which are extensions of T by the full infinite braid group, not just of their relative abelianizations T^*_{ab} and $T^\sharp_{ab}$. The core question here is how to faithfully represent the braids.

The representation-theoretic interpretation [FrKi] of the Kashaev operators may suggest a possible solution to this problem. As suggested by Frenkel [Fr], one natural candidate is the 'braiding', or R-martrix, in terms of the representation theory of the (e.g. quasitriangular) Hopf algebras:

$$R_{12} : \mathscr{H}_1 \otimes \mathscr{H}_2 \to \mathscr{H}_2 \otimes \mathscr{H}_1. \tag{6.28}$$

Nevertheless, unlike its 'Drinfeld double' $\mathcal{U}_{q\widetilde{q}}(\mathfrak{sl}(2,\mathbb{R}))$, the (modular double of the) quantum plane algebra $B_{q\widetilde{q}}$ is not yet known to be quasitriangular, so there is no obvious R-matrix. One can look for some additional structure on $B_{q\widetilde{q}}$, which may not be a quasitriangular structure but still leads to what would play the role of an R-matrix.

Meanwhile, Lochak and Schneps [LoSc] considered attaching a flat ribbon along each ideal arc, and studied braiding of those flat ribbons. This provides a certain 'extension' of the Ptolemy groupoid by the braid group, which we expect to be related to the group T^* (see also [Br] [D] for the group BV considered by Brin and Dehornoy, and [FuKap1] for 'the universal mapping

class group in genus zero $\mathcal{B}$'; and see e.g. the survey [FuKapS] or the introduction of [FuKap2]), and Lochak-Schneps showed that the Grothendiek-Teichmüller group $\widehat{GT}$, and therefore also one of its famous subgroup $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galoids group, act on this groupoid. Hence, if one can interpret these flat ribbons in terms of the representation theory of $B_{q\widetilde{q}}$ (i.e. at the level of $\mathscr{H}$), then this may lead to the construction of an action of the absolute Galois group on the representation-theoretic version of the quantum universal Teichmüller space (which is a question raised by Frenkel [Fr]), which is of immense interest to the author.

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